BASIC ELECTROMAGNETICS WITH APPLICATIONS

by

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Solution Manual

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Pearson Prentice Hall, Upper Saddle River, NJ, USA
March 17, 2009

CHAPTER 1

1.1. From graphical constructions similar to those in Example 1-1, we obtain the answers given on page 529 of the text.

1.2. (a) Area of the triangle $= \frac{1}{2} AB \sin \alpha_{\underset{\sim}{A}}^{\underset{\sim}{B}} = \frac{1}{2} |\underset{\sim}{A} \times \underset{\sim}{B}|$.

(b) Volume of the tetrahedron $= \frac{1}{3}$ area of one face x length of the perpendicular drawn to that face from the vertex opposite to it

$$= \left(\frac{1}{3}\right)\left(\frac{1}{2}|\underset{\sim}{B} \times \underset{\sim}{C}|\right)\left(\frac{|\underset{\sim}{A} \cdot \underset{\sim}{B} \times \underset{\sim}{C}|}{|\underset{\sim}{B} \times \underset{\sim}{C}|}\right) = \frac{1}{6} |\underset{\sim}{A} \cdot \underset{\sim}{B} \times \underset{\sim}{C}|.$$

1.3. $\underset{\sim}{C} \cdot \underset{\sim}{C} = (\underset{\sim}{B} - \underset{\sim}{A}) \cdot (\underset{\sim}{B} - \underset{\sim}{A}) = \underset{\sim}{B} \cdot \underset{\sim}{B} - \underset{\sim}{B} \cdot \underset{\sim}{A} - \underset{\sim}{A} \cdot \underset{\sim}{B} + \underset{\sim}{A} \cdot \underset{\sim}{A}$

$$c^2 = \underset{\sim}{B} \cdot \underset{\sim}{B} - 2\underset{\sim}{A} \cdot \underset{\sim}{B} + \underset{\sim}{A} \cdot \underset{\sim}{A} = A^2 + B^2 - 2AB \cos \alpha_{\underset{\sim}{A}}^{\underset{\sim}{B}}.$$

1.4. (a) A normal vector to the plane $= (\underset{\sim}{A} - \underset{\sim}{B}) \times (\underset{\sim}{A} - \underset{\sim}{C})$

$$= \underset{\sim}{A} \times \underset{\sim}{A} - \underset{\sim}{A} \times \underset{\sim}{C} - \underset{\sim}{B} \times \underset{\sim}{A} + \underset{\sim}{B} \times \underset{\sim}{C} = \underset{\sim}{A} \times \underset{\sim}{B} + \underset{\sim}{B} \times \underset{\sim}{C} + \underset{\sim}{C} \times \underset{\sim}{A}.$$

(b) Minimum distance is the component of $\underset{\sim}{A}$ (or $\underset{\sim}{B}$ or $\underset{\sim}{C}$) along the normal vector to the plane from the origin

$$= \frac{|\underset{\sim}{A} \cdot [(\underset{\sim}{A} - \underset{\sim}{B}) \times (\underset{\sim}{A} - \underset{\sim}{C})]|}{|(\underset{\sim}{A} - \underset{\sim}{B}) \times (\underset{\sim}{A} - \underset{\sim}{C})|}.$$

1.5. (a) $d\underset{\sim}{l} = dx\, \underset{\sim}{i}_x + dy\, \underset{\sim}{i}_y + dz\, \underset{\sim}{i}_z$

$$= (\cos\phi\, dr - r \sin\phi\, d\phi)\, \underset{\sim}{i}_x + (\sin\phi\, dr + r\cos\phi\, d\phi)\, \underset{\sim}{i}_y + dz\, \underset{\sim}{i}_z$$

$$= (\cos\phi\, \underset{\sim}{i}_x + \sin\phi\, \underset{\sim}{i}_y)\, dr + (r\cos\phi\, \underset{\sim}{i}_y - r\sin\phi\, \underset{\sim}{i}_x)\, d\phi + dz\, \underset{\sim}{i}_z$$

$$= dr\, \underset{\sim}{i}_r + r\, d\phi\, \underset{\sim}{i}_\phi + dz\, \underset{\sim}{i}_z.$$

(b) Derivation similar to that in part (a).

(c) $dl = [(u\, du - v\, dv)^2 + (u\, dv + v\, du)^2 + (dz)^2]^{1/2}$

$$= [(u^2+v^2)(du)^2 + (u^2+v^2)(dv)^2 + (dz)^2]^{1/2}$$

$$d\underset{\sim}{l} = \sqrt{u^2+v^2}\, du\, \underset{\sim}{i}_u + \sqrt{u^2+v^2}\, dv\, \underset{\sim}{i}_v + dz\, \underset{\sim}{i}_z$$

(d) $dv = (\sqrt{u^2+v^2}\, du)(\sqrt{u^2+v^2}\, dv)(dz) = (u^2+v^2)\, du\, dv\, dz.$

1.6. For method, see Example 1-4.

1.7. (a) $\cos \eta = \dfrac{\mathbf{OT} \cdot \mathbf{OR}}{(OT)(OR)}$

(b) $\cos \alpha = \dfrac{\mathbf{OT} \times \mathbf{OR}}{(OT)(OR)\sin\eta} \cdot \dfrac{\mathbf{OT} \times \mathbf{ON}}{(OT)(ON)\sin\theta_T}$

(c) and (d) For answers, see page 529 of text.

1.8. For method, see Example 1-5.

1.9. For method, see Example 1-6.

1.10. (a) The two vectors are equal since $\mathbf{i}_x$, $\mathbf{i}_y$, and $\mathbf{i}_z$ are uniform.

(b) $$[\mathbf{i}_r + \mathbf{i}_\phi + 3\,\mathbf{i}_z]_{(2,\frac{\pi}{2},3)} = \left(\cos\frac{\pi}{2} - \sin\frac{\pi}{2}\right)\mathbf{i}_x + \left(\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right)\mathbf{i}_y + 3\,\mathbf{i}_z = -\mathbf{i}_x + \mathbf{i}_y + 3\,\mathbf{i}_z$$

$$[\mathbf{i}_r + \mathbf{i}_\phi + 3\,\mathbf{i}_z]_{(3.6,\,3\frac{\pi}{4},\,9.4)} = \left(\cos\frac{3\pi}{4} - \sin\frac{3\pi}{4}\right)\mathbf{i}_x + \left(\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4}\right)\mathbf{i}_y + 3\,\mathbf{i}_z = -\sqrt{2}\,\mathbf{i}_x + 3\,\mathbf{i}_z$$

The two vectors are not equal.

(c) $$[\sqrt{2}\,\mathbf{i}_r + 3\,\mathbf{i}_z]_{(3.6,\,\frac{3\pi}{4},\,9.4)} = \sqrt{2}\cos\frac{3\pi}{4}\,\mathbf{i}_x + \sqrt{2}\sin\frac{3\pi}{4}\,\mathbf{i}_y + 3\,\mathbf{i}_z = -\mathbf{i}_x + \mathbf{i}_y + 3\,\mathbf{i}_z.$$

The two vectors are equal.

(d) not equal

(e) equal

1.11. (a) $$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = [(\mathbf{C}\times\mathbf{D})\times\mathbf{A}]\cdot\mathbf{B} = [(\mathbf{C}\cdot\mathbf{A})\mathbf{D} - (\mathbf{D}\cdot\mathbf{A})\mathbf{C}]\cdot\mathbf{B}$$
$$= (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{B}\cdot\mathbf{C})(\mathbf{A}\cdot\mathbf{D})$$

(b) $$(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = \mathbf{E}\times(\mathbf{C}\times\mathbf{D}) = (\mathbf{E}\cdot\mathbf{D})\mathbf{C} - (\mathbf{E}\cdot\mathbf{C})\mathbf{D}$$
$$= (\mathbf{A}\times\mathbf{B}\cdot\mathbf{D})\mathbf{C} - (\mathbf{A}\times\mathbf{B}\cdot\mathbf{C})\mathbf{D}$$

(c) $$(\mathbf{B}\times\mathbf{C})\times(\mathbf{C}\times\mathbf{A}) = (\mathbf{B}\times\mathbf{C}\cdot\mathbf{A})\mathbf{C} - (\mathbf{B}\times\mathbf{C}\cdot\mathbf{C})\mathbf{A} = (\mathbf{B}\times\mathbf{C}\cdot\mathbf{A})\mathbf{C}$$
$$(\mathbf{A}\times\mathbf{B})\times(\mathbf{B}\times\mathbf{C})\times(\mathbf{C}\times\mathbf{A}) = \mathbf{A}\times\mathbf{B}\cdot(\mathbf{B}\times\mathbf{C}\cdot\mathbf{A})\mathbf{C} = (\mathbf{A}\times\mathbf{B}\cdot\mathbf{C})^2$$

(d) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) + \mathbf{B}\times(\mathbf{C}\times\mathbf{A}) + \mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{A}\cdot\mathbf{C})\,\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\,\mathbf{C}$

$+ (\mathbf{B}\cdot\mathbf{A})\,\mathbf{C} - (\mathbf{B}\cdot\mathbf{C})\,\mathbf{A} + (\mathbf{C}\cdot\mathbf{B})\,\mathbf{A} - (\mathbf{C}\cdot\mathbf{A})\,\mathbf{B} = 0.$

1.12. (a) $3\,\mathbf{i}_x + 6\,\mathbf{i}_y + 3\,\mathbf{i}_z$, $2\,\mathbf{i}_x - 5\,\mathbf{i}_y - 2\,\mathbf{i}_z$, $3\,\mathbf{i}_x + 3\,\mathbf{i}_y + 5\,\mathbf{i}_z$

(b) $11\,\mathbf{i}_x - 2\,\mathbf{i}_y + 9\,\mathbf{i}_z$, $-6\,\mathbf{i}_x + 10\,\mathbf{i}_y$

(c) $\sqrt{19}$, $\sqrt{33}$

(d) $(2\,\mathbf{i}_x - 5\,\mathbf{i}_y - 2\,\mathbf{i}_z)/\sqrt{33}$

(e) 10, 0, −6

(f) $5/7$, $44^\circ 25'$; 0, 90°; -0.2809, $106^\circ 19'$

(g) $-4\,\mathbf{i}_x + 8\,\mathbf{i}_y - 4\,\mathbf{i}_z$, $4\,\mathbf{i}_x - 2\,\mathbf{i}_y - 8\,\mathbf{i}_z$, $-8\,\mathbf{i}_x - 2\,\mathbf{i}_y + 4\,\mathbf{i}_z$,

$-10\,\mathbf{i}_x + 20\,\mathbf{i}_y - 10\,\mathbf{i}_z$, $10\,\mathbf{i}_x - 20\,\mathbf{i}_y + 10\,\mathbf{i}_z$, 0

(h) $5/7$, $134^\circ 25'$; $5/7$, $134^\circ 25'$; 0, 180°

(i) 0

(j) 16, −24, −24, −24

(k) $-32\,\mathbf{i}_x - 8\,\mathbf{i}_y + 16\,\mathbf{i}_z$

(l) 576

(m) 0

(n) $(\cos\phi - 2\sin\phi)\,\mathbf{i}_{rc} + (-\sin\phi - 2\cos\phi)\,\mathbf{i}_\phi + \mathbf{i}_z$;

$(\sin\theta\cos\phi - 2\sin\theta\sin\phi + \cos\theta)\,\mathbf{i}_{rs} + (\cos\theta\cos\phi - 2\cos\theta\sin\phi$

$-\sin\theta)\,\mathbf{i}_\theta + (-\sin\phi - 2\cos\phi)\,\mathbf{i}_\phi$

(o) $(\mathbf{A}+\mathbf{B})\times\mathbf{i}_x = -4\,\mathbf{i}_z + 4\,\mathbf{i}_y$ is a vector perpendicular to $\mathbf{A}+\mathbf{B}$.

1.13. For $\mathbf{A}$ and $\mathbf{B}$ in the first quadrant, $\mathbf{A} = A(\cos\alpha\,\mathbf{i}_x + \sin\alpha\,\mathbf{i}_y)$ and $\mathbf{B} = B(\cos\beta\,\mathbf{i}_x + \sin\beta\,\mathbf{i}_y)$. Then

(a) $\cos(\alpha-\beta) = \dfrac{\mathbf{A}\cdot\mathbf{B}}{AB} = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

(b) $\sin(\alpha-\beta) = \dfrac{|\mathbf{A}\times\mathbf{B}|}{AB} = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

For $\mathbf{A}$ in the first quadrant and $\mathbf{B}$ in the fourth quadrant,

$\mathbf{A} = A(\cos\alpha\,\mathbf{i}_x + \sin\alpha\,\mathbf{i}_y)$ and $\mathbf{B} = B(\cos\beta\,\mathbf{i}_x - \sin\beta\,\mathbf{i}_y)$. Then

(c) $\cos(\alpha+\beta) = \dfrac{\mathbf{A}\cdot\mathbf{B}}{AB} = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

(d) $\sin(\alpha+\beta) = \dfrac{|\underset{\sim}{A}\times\underset{\sim}{B}|}{AB} = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

1.14. Component of $\underset{\sim}{A}$ along $\underset{\sim}{B}$ $= A\cos\alpha\angle_{\underset{\sim}{B}}^{\underset{\sim}{A}} = A\,\dfrac{\underset{\sim}{A}\cdot\underset{\sim}{B}}{AB} = \dfrac{\underset{\sim}{A}\cdot\underset{\sim}{B}}{B}$; 1.871

1.15. Noting that the intercepts made on the x, y, and z axes by the plane are 3, 1.5, and 1, respectively, we can write two vectors in the plane as $\underset{\sim}{A} = -3\,\underset{\sim}{i}_x + 1.5\,\underset{\sim}{i}_y$ and $\underset{\sim}{B} = -3\,\underset{\sim}{i}_x + \underset{\sim}{i}_z$. Unit vector normal to the plane is then given by $\dfrac{\underset{\sim}{A}\cdot\underset{\sim}{B}}{AB} = \dfrac{1}{\sqrt{14}}(\underset{\sim}{i}_x + 2\,\underset{\sim}{i}_y + 3\,\underset{\sim}{i}_z)$.

1.16. Vector drawn from (x_0, y_0, z_0) to an arbitrary point (x, y, z) in the plane is $(x-x_0)\,\underset{\sim}{i}_x + (y-y_0)\,\underset{\sim}{i}_y + (z-z_0)\,\underset{\sim}{i}_z$. This vector must be perpendicular to $a\,\underset{\sim}{i}_x + b\,\underset{\sim}{i}_y + c\,\underset{\sim}{i}_z$ for all (x, y, z) on the plane. Hence, the equation of the plane is given by

$$(a\,\underset{\sim}{i}_x + b\,\underset{\sim}{i}_y + c\,\underset{\sim}{i}_z)\cdot[(x-x_0)\,\underset{\sim}{i}_x + (y-y_0)\,\underset{\sim}{i}_y + (z-z_0)\,\underset{\sim}{i}_z] = 0$$

or, $$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

1.17. For descriptions, see page 529 of the text.

(a) $x^2 + 4y^2 + 9z^2 = T_0$ or, $\left(\dfrac{x}{\sqrt{T_0}}\right)^2 + \left(\dfrac{y}{\sqrt{T_0/2}}\right)^2 + \left(\dfrac{z}{\sqrt{T_0}/3}\right)^2 = 1$

(b) $\dfrac{\cos\phi}{r} = U_0$ or, $r = \dfrac{1}{U_0}\cos\phi$ or, $r^2 = \dfrac{x}{U_0}$ since $x = r\cos\phi$

$x^2 + y^2 = \dfrac{x}{U_0}$ or, $\left(x - \dfrac{1}{2U_0}\right)^2 + y^2 = \left(\dfrac{1}{2U_0}\right)^2$

(c) $\dfrac{\sin\theta}{r_s} = V_0$ or, $r_s = \dfrac{1}{V_0}\sin\theta$ or, $r_s^2 = \dfrac{r_c}{V_0}$ since $r_c = r_s\sin\theta$

$r_c^2 + z^2 = \dfrac{r_c}{V_0}$ or, $\left(r_c - \dfrac{1}{2V_0}\right)^2 + z^2 = \left(\dfrac{1}{2V_0}\right)^2$

1.18. $\underset{\sim}{v} = \omega r\sin\theta\,\underset{\sim}{i}_\phi$ where ω is the angular velocity of the earth. Constant magnitude surfaces are $r\sin\theta =$ constant, cylinders having the z axis as the axis. Direction lines are circles in the planes $z =$ constant and with their centers located along the z axis, i.e., the spin axis.

1.19. See page 529 of the text.

1.20. (a) Constant magnitude surfaces are $x =$ constant. Direction lines are straight lines directed in the positive x direction for $x > 2$ and in the negative x direction for $x < 2$.

(b) Constant magnitude surfaces are $r =$ constant. Direction lines are circles in the planes $z =$ constant with centers along the z axis and directed in the positive ϕ direction for $r > 1$ but in the negative ϕ direction for $r < 1$.

(c) Constant magnitude surfaces are $r =$ constant. Direction lines are semicircles in the $\phi =$ constant planes with centers at the origin and directed in the θ direction.

(d) Constant magnitude surfaces are $r =$ constant. Direction lines are radial lines emanating from the origin.

1.21. For method, see Example 1-10.

1.22. In cartesian coordinates, $\underset{\sim}{v} = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i}_x + \frac{dy}{dt}\underset{\sim}{i}_y + \frac{dz}{dt}\underset{\sim}{i}_z$, and

$$\underset{\sim}{a} = \frac{d\underset{\sim}{v}}{dt} = \frac{d^2x}{dt^2}\underset{\sim}{i}_x + \frac{d^2y}{dt^2}\underset{\sim}{i}_y + \frac{d^2z}{dt^2}\underset{\sim}{i}_z.$$

In cylindrical coordinates, $\underset{\sim}{v} = \frac{d\underset{\sim}{r}}{dt} = \frac{dr_c}{dt}\underset{\sim}{i}_{rc} + r_c\frac{d\underset{\sim}{i}_{rc}}{dt} + \frac{dz}{dt}\underset{\sim}{i}_z$

But $\frac{d\underset{\sim}{i}_{rc}}{dt} = \frac{\partial \underset{\sim}{i}_{rc}}{\partial r_c}\frac{dr_c}{dt} + \frac{\partial \underset{\sim}{i}_{rc}}{\partial \phi}\frac{d\phi}{dt} + \frac{\partial \underset{\sim}{i}_{rc}}{\partial z}\frac{dz}{dt} = \underset{\sim}{i}_\phi \frac{d\phi}{dt}$.

$$\therefore \underset{\sim}{v} = \frac{dr_c}{dt}\underset{\sim}{i}_{rc} + r_c\frac{d\phi}{dt}\underset{\sim}{i}_\phi + \frac{dz}{dt}\underset{\sim}{i}_z.$$

$$\underset{\sim}{a} = \frac{d\underset{\sim}{v}}{dt} = \frac{d^2r_c}{dt^2}\underset{\sim}{i}_{rc} + \frac{dr_c}{dt}\frac{d\underset{\sim}{i}_{rc}}{dt} + \frac{dr_c}{dt}\frac{d\phi}{dt}\underset{\sim}{i}_\phi + r_c\frac{d^2\phi}{dt^2}\underset{\sim}{i}_\phi + r_c\frac{d\phi}{dt}\frac{d\underset{\sim}{i}_\phi}{dt} + \frac{d^2z}{dt^2}\underset{\sim}{i}_z$$

But $\frac{d\underset{\sim}{i}_\phi}{dt} = \frac{\partial \underset{\sim}{i}_\phi}{\partial r_c}\frac{dr_c}{dt} + \frac{\partial \underset{\sim}{i}_\phi}{\partial \phi}\frac{d\phi}{dt} + \frac{\partial \underset{\sim}{i}_\phi}{\partial z}\frac{dz}{dt} = -\underset{\sim}{i}_{rc}\frac{d\phi}{dt}$.

$$\therefore \underset{\sim}{a} = \left[\frac{d^2r_c}{dt^2} - r_c\left(\frac{d\phi}{dt}\right)^2\right]\underset{\sim}{i}_r + \frac{1}{r_c}\frac{d}{dt}\left(r_c^2\frac{d\phi}{dt}\right)\underset{\sim}{i}_\phi + \frac{d^2z}{dt^2}\underset{\sim}{i}_z.$$

Similarly, in spherical coordinates,

$$\underset{\sim}{v} = \frac{dr_s}{dt}\underset{\sim}{i}_{rs} + r_s\frac{d\theta}{dt}\underset{\sim}{i}_\theta + r_s\sin\theta\frac{d\phi}{dt}\underset{\sim}{i}_\phi$$

$$\mathbf{a} = \left[\frac{d^2 r_s}{dt^2} - r_s \left(\frac{d\theta}{dt} \right)^2 - r_s \left(\sin\theta \, \frac{d\phi}{dt} \right)^2 \right] \mathbf{i}_{rs}$$

$$+ \left[\frac{1}{r_s} \frac{d}{dt} \left(r_s^2 \frac{d\theta}{dt} \right) - r_s \sin\theta \cos\theta \left(\frac{d\phi}{dt} \right)^2 \right] \mathbf{i}_\phi$$

$$+ \left[\frac{1}{r_s \sin\theta} \frac{d}{dt} \left(r_s^2 \sin^2\theta \, \frac{d\phi}{dt} \right) \right] \mathbf{i}_\phi$$

1.23. Using expressions for $\mathbf{v}$ and $\mathbf{a}$ given in the solutions for Problem 1.22, we get the answers given on page 529 of the text.

1.24. $d(\mathbf{A} \cdot \mathbf{B}) = d(A_x B_x + A_y B_y + A_z B_z)$

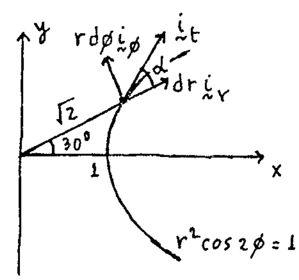
$$= dA_x \, B_x + dA_y \, B_y + dA_z \, B_z + A_x \, dB_x + A_y \, dB_y + A_z \, dB_z$$

$$= d\mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot d\mathbf{B}$$

Equation (1-64) can be verified in a similar manner.

1.25. (a) $d(r^2 \cos 2\phi) = 2r \cos 2\phi \, dr - 2r^2 \sin 2\phi = 0$ on the surface. Hence,

$$\tan\alpha = \left(\frac{r \, d\phi}{dr} \right)_{\sqrt{2}, \frac{\pi}{6}, 0} = (\cot 2\phi)_{\frac{\pi}{6}} = \frac{1}{\sqrt{3}} \quad ; \alpha = 30^\circ$$



$$\mathbf{i}_t = \cos\alpha \, \mathbf{i}_r + \sin\alpha \, \mathbf{i}_\phi = \frac{\sqrt{3}}{2} \mathbf{i}_r + \frac{1}{2} \mathbf{i}_\phi$$

Using $\mathbf{i}_z$ as the second vector tangential to the surface, we obtain

$$\mathbf{i}_n = \mathbf{i}_t \times \mathbf{i}_z = \frac{1}{2} \mathbf{i}_r - \frac{\sqrt{3}}{2} \mathbf{i}_\phi .$$

(b) $$[\nabla (r^2 \cos 2\phi)]_{\sqrt{2}, \frac{\pi}{6}, 0} = [2r \cos 2\phi \, \mathbf{i}_r - 2r \sin 2\phi \, \mathbf{i}_\phi]_{\sqrt{2}, \frac{\pi}{6}, 0}$$

$$= 2\sqrt{2} \cos \frac{\pi}{3} \mathbf{i}_r - 2\sqrt{2} \sin \frac{\pi}{3} \mathbf{i}_\phi$$

$$\therefore \mathbf{i}_n = \cos \frac{\pi}{3} \mathbf{i}_r - \sin \frac{\pi}{3} \mathbf{i}_\phi = \frac{1}{2} \mathbf{i}_r - \frac{\sqrt{3}}{2} \mathbf{i}_\phi .$$

1.26. (a) $dT = \nabla T \cdot d\mathbf{l} = (yz \, \mathbf{i}_x + zx \, \mathbf{i}_y + xy \, \mathbf{i}_z) \cdot (dx \, \mathbf{i}_x + dy \, \mathbf{i}_y + dz \, \mathbf{i}_z)$

$$= yz \, dx + zx \, dy + xy \, dz = d(xyz)$$

$$\therefore T = xyz$$

(b) $U = x^3 y z^2$

(c) $V = -\frac{1}{r} \cos\phi$

(d) $\quad W = r^{-n}$

1.27. See page 530 of the text.

1.28. Unit vector normal to the surface at $(2,1,1) = \left(\dfrac{\nabla V}{|\nabla V|}\right)_{2,1,1} = \dfrac{2\,\mathbf{i}_x - \mathbf{i}_y}{\sqrt{5}}$

Vector joining $(1,-2,0)$ to $(0,0,2) = -\mathbf{i}_x + 2\,\mathbf{i}_y + 2\,\mathbf{i}_z$

Required component $= \dfrac{2\,\mathbf{i}_x - \mathbf{i}_y}{\sqrt{5}} \cdot \dfrac{-\mathbf{i}_x + 2\,\mathbf{i}_y + 2\,\mathbf{i}_z}{3} = -\dfrac{4}{3\sqrt{5}}$.

1.29. The required rate of change is

$$[\nabla(x^2y + yz^2 + zy^2)]_{(1,2,3)} \cdot \frac{[\nabla(x^2y - yz + xz^2)]_{(1,2,3)}}{|[\nabla(x^2y - yz + xz^2)]_{(1,2,3)}|}$$

$$= (4\,\mathbf{i}_x + 22\,\mathbf{i}_y + 16\,\mathbf{i}_z) \cdot \frac{13\,\mathbf{i}_x - 2\,\mathbf{i}_y + 4\,\mathbf{i}_z}{|13\,\mathbf{i}_x - 2\,\mathbf{i}_y + 4\,\mathbf{i}_z|} = 5.237$$

1.30. A normal vector to the plane at $(\frac{1}{2}, \frac{1}{4}, 8)$ is $[\nabla(xyz)]_{\frac{1}{2}, \frac{1}{4}, 8}$

$= 2\,\mathbf{i}_x + 4\,\mathbf{i}_y + \frac{1}{8}\,\mathbf{i}_z$. From the solution to Problem 1.6., the required plane is given by $2(x - \frac{1}{2}) + 4(y - \frac{1}{4}) + \frac{1}{8}(z - 8) = 0$

or, $\quad 16x + 32y + z = 24$.

1.31. (a) $\displaystyle\int_V xyz\, dv = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} xyz\, dx\, dy\, dz = \frac{1}{720}$.

(b) $\displaystyle\int_V \frac{1}{r}\, dv = \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \int_{z=0}^{l} \frac{1}{r}\, r\, dr\, d\phi\, dz = 2\pi a l$.

(c) $\displaystyle\int_V x\, dv = \int_{r=0}^{1} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} r\sin\theta\cos\phi \cdot r^2 \sin\theta\, dr\, d\theta\, d\phi = \frac{\pi}{16}$.

1.32. (a) $\displaystyle\oint \mathbf{A} \cdot d\mathbf{S} = \int_{x=0}^{1}\int_{y=0}^{1} (0)\, dx\, dy + \int_{y=0}^{1}\int_{z=0}^{1} (0)\, dy\, dz + \int_{z=0}^{1}\int_{x=0}^{1} (0)\, dz\, dx$

$$+ \int_{x=0}^{1}\int_{y=0}^{1} xy\, dx\, dy + \int_{y=0}^{1}\int_{z=0}^{1} yz\, dy\, dz + \int_{z=0}^{1}\int_{x=0}^{1} xz\, dz\, dx = \frac{3}{4}.$$

(b) $\mathbf{A} \cdot d\mathbf{S}$ exists only for the surface $x + 2y + 3z = 3$. For this surface,

$$\mathbf{A} \cdot d\mathbf{S} = \mathbf{A} \cdot \frac{dx\, dy}{\mathbf{i}_n \cdot \mathbf{i}_z}\, \mathbf{i}_n$$

$$= (x^2yz\,\mathbf{i}_x + y^2zx\,\mathbf{i}_y + z^2xy\,\mathbf{i}_z) \cdot \left(\tfrac{1}{3}\,\mathbf{i}_x + \tfrac{2}{3}\,\mathbf{i}_y + \mathbf{i}_z\right) dx\, dy$$

$$= x y z \, dx \, dy$$

$$\oint \underset{\sim}{A} \cdot d\underset{\sim}{s} = \int_{x=0}^{3} \int_{y=0}^{\frac{3-x}{2}} xy \left(\frac{3-x-2y}{3} \right) dx \, dy = \frac{27}{160}.$$

1.33. (a) $$\oint \underset{\sim}{A} \cdot d\underset{\sim}{s} = \int_{r=0}^{a} \int_{\phi=0}^{2\pi} (0) \, r \, dr \, d\phi + \int_{r=0}^{a} \int_{\phi=0}^{2\pi} (0) \, r \, dr \, d\phi + \int_{\phi=0}^{2\pi} \int_{z=0}^{l} a^2 \cos\phi \, d\phi \, dz = 0$$

(b) $$\oint \underset{\sim}{A} \cdot d\underset{\sim}{s} = \int_{r=0}^{a} \int_{\phi=0}^{2\pi} (0) \, r \, dr \, d\phi + \int_{r=0}^{a} \int_{\phi=0}^{2\pi} (0) \, r \, dr \, d\phi + \int_{r=0}^{a} \int_{z=0}^{l} (0) \, dr \, dz + \int_{r=0}^{a} \int_{z=0}^{l} -r \, dr \, dz + \int_{\phi=0}^{\pi/2} \int_{z=0}^{l} a^2 \cos\phi \, d\phi \, dz = a^2 \frac{l}{2}.$$

1.34. (a) $$\oint \underset{\sim}{A} \cdot d\underset{\sim}{s} = \int_{r=0}^{1} \int_{\theta=0}^{\pi/2} (0) \, r \, dr \, d\theta + \int_{r=0}^{1} \int_{\theta=0}^{\pi/2} (0) \, r \, dr \, d\theta + \int_{r=0}^{1} \int_{\phi=0}^{\pi/2} r^2 \, dr \, d\phi + \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \sin\theta \, d\theta \, d\phi = \frac{2\pi}{3}.$$

(b) $$\oint \underset{\sim}{A} \cdot d\underset{\sim}{s} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} -a^4 \sin\theta \, d\theta \, d\phi + \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} b^4 \sin\theta \, d\theta \, d\phi = 4\pi(b^4 - a^4).$$

1.35. (a) $$\int_{0,0,0}^{\pi/2,1,0} \underset{\sim}{F} \cdot d\underset{\sim}{l} = \int_{x=0}^{\pi/2} (\sin^2 x \, dx + x \sin 2x \, dx) = \frac{\pi}{2}.$$

(b) $$\int_{0,0,0}^{\pi/2,1,0} \underset{\sim}{F} \cdot d\underset{\sim}{l} = \int_{x=0}^{\pi/2} \left(\frac{4}{\pi^2} x^2 \, dx + \frac{8}{\pi^2} x^2 \, dx \right) = \frac{\pi}{2}.$$

(c) $$\int_{0,0,0}^{\pi/2,1,0} \underset{\sim}{F} \cdot d\underset{\sim}{l} = \int_{y=0}^{1} \left(\pi y^2 \, dy + \frac{\pi}{2} y^2 \, dy \right) = \frac{\pi}{2}.$$

(d) For any arbitrary path, $\int_{0,0,0}^{\pi/2,1,0} \underset{\sim}{F} \cdot d\underset{\sim}{l} = \int_{0,0,0}^{\pi/2,1,0} (y \, dx + x \, dy) = \int_{0,0,0}^{\pi/2,1,0} d(xy) = \frac{\pi}{2}.$

1.36. (a) $$\int_{0,0,0}^{1,1,1} \underset{\sim}{A} \cdot d\underset{\sim}{l} = \int_{z=0}^{1} 2(a^2 - b^2) z^3 \, dz = \frac{a^2 - b^2}{2}.$$

(b) $$\int_{0,0,0}^{1,1,1} \underset{\sim}{A} \cdot d\underset{\sim}{l} = \int_{x=0}^{1} (0) \, dx + \int_{y=0}^{1} -b^2 \, dy + \int_{z=0}^{1} (0) \, dz = -b^2$$

(c) $$\int_{0,0,0}^{1,1,1} \underset{\sim}{A} \cdot d\underset{\sim}{l} = \int_{x=0}^{1} (a^2 - b^2) \, x \, dx + \int_{z=0}^{1} (0) \, dz = \frac{a^2 - b^2}{2}$$

(d) $$\int_{0,0,0}^{1,1,1} \underset{\sim}{A} \cdot d\underset{\sim}{l} = \int_{y=0}^{1} (0) \, dy + \int_{x=0}^{1} a^2 \, dx + \int_{z=0}^{1} (0) \, dz = a^2$$

(e) $\displaystyle\int_{0,0,0}^{1,1,1} \mathbf{A}\cdot d\mathbf{l} = \int_{x=0}^{1} (a^2-b^2)\, x\, dx = \frac{a^2-b^2}{2}$. In fact, for any path

in the xy plane, $\displaystyle\int_{0,0,0}^{1,1,1} \mathbf{A}\cdot d\mathbf{l} = \int_{0,0,0}^{1,1,1} d\left(\frac{a^2x^2}{2} - \frac{b^2y^2}{2}\right) = \frac{a^2-b^2}{2}$.

1.37. $\displaystyle\oint_{abcda} \mathbf{A}\cdot d\mathbf{l} = 0 + 0 + \int_{x=1}^{0} x^2\, dx + \int_{y=1}^{0} y^2\, dy = -\frac{2}{3}$.

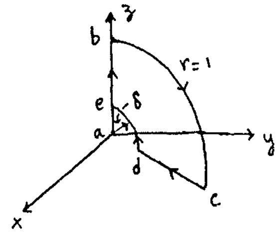
1.38. (a) $\displaystyle\oint_C \mathbf{A}\cdot d\mathbf{l} = \int_{r=0}^{1} 2r\, dr + \int_{\phi=0}^{\pi/2} d\phi + 0 = 1 + 0.5\pi$

(b) $\displaystyle\oint_{C_1} \mathbf{A}\cdot d\mathbf{l} + \oint_{C_2} \mathbf{A}\cdot d\mathbf{l} = \int_{\phi=0}^{2\pi} b^2\, d\phi + \int_{\phi=0}^{-2\pi} a^2\, d\phi = 2\pi(b^2-a^2)$.

1.39. Let us consider the contour bcdeb shown in the figure. Then

$$\oint_{ebcde} \mathbf{A}\cdot d\mathbf{l} = 0 + \int_{\theta=0}^{\pi/2} e^{-1}\, d\theta + 0 + \int_{\theta=\frac{\pi}{2}}^{0} e^{-\delta}\, d\theta$$

$$= \frac{\pi}{2}\left(e^{-1} - e^{-\delta}\right)$$



$$\oint_{abca} \mathbf{A}\cdot d\mathbf{l} = \lim_{\delta\to 0} \oint_{ebcde} \mathbf{A}\cdot d\mathbf{l} = \frac{\pi}{2}\left(e^{-1}-1\right).$$

1.40. (a) $\displaystyle\oint_C d\mathbf{l} = \oint_C (dx\, \mathbf{i}_x + dy\, \mathbf{i}_y + dz\, \mathbf{i}_z) = \left(\oint_C dx\right)\mathbf{i}_x + \left(\oint_C dy\right)\mathbf{i}_y + \left(\oint_C dz\right)\mathbf{i}_z = 0$

(b) $\displaystyle\oint_S d\mathbf{S} = \int_{\substack{\text{flat}\\\text{surface}}} d\mathbf{S} + \int_{\substack{\text{hemispherical}\\\text{surface}}} d\mathbf{S}$

$$= \int_{r=0}^{a}\int_{\phi=0}^{2\pi} -r\, dr\, d\phi\, \mathbf{i}_z + \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} a^2 \sin\theta\, d\theta\, d\phi\, \mathbf{i}_r$$

$$= -\left[\int_{r=0}^{a}\int_{\phi=0}^{2\pi} dr\, d\phi\right]\mathbf{i}_z + \left[\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} a^2 \sin^2\theta \cos\phi\, d\theta\, d\phi\right]\mathbf{i}_x$$

$$+ \left[\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} a^2 \sin^2\theta \sin\phi\, d\theta\, d\phi\right]\mathbf{i}_y + \left[\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} a^2 \sin\theta\cos\theta\, d\theta\, d\phi\right]\mathbf{i}_z = 0$$

(c) $\displaystyle\int_V \mathbf{i}_\theta\, dv = \int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} (\cos\theta\cos\phi\, \mathbf{i}_x + \cos\theta\sin\phi\, \mathbf{i}_y - \sin\theta\, \mathbf{i}_z)\, r^2 \sin\theta\, dr\, d\theta\, d\phi$

$$= -\frac{1}{3}\pi^2 a^3\, \mathbf{i}_z.$$

1.41. Derivation similar to that in cylindrical coordinates (See section 1.8)

1.42. Derivation similar to that in cylindrical coordinates (See section 1.8)

1.43. See page 530 of the text.

1.44. (a) $6xyz$ (b) 5 (c) $\cos\phi$ (d) 0 except at the origin (e) $4r + 2\cos\theta$.

1.45. $\oint_S \underset{\sim}{r} \cdot d\underset{\sim}{s} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} a\, \underset{\sim}{i}_r \cdot a^2 \sin\theta \, d\theta \, d\phi = 4\pi a^3.$

$\int_V \underset{\sim}{\nabla} \cdot \underset{\sim}{r}\, dv = \int_V 3\, dv = 3 \times \frac{4}{3}\pi a^3 = 4\pi a^3.$

Hence, divergence theorem is verified.

1.46. (a) $\int_V \underset{\sim}{\nabla} \cdot \underset{\sim}{A}\, dv = \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} 6xyz\, dx\, dy\, dz = \frac{3}{4}.$

(b) $\int_V \underset{\sim}{\nabla} \cdot \underset{\sim}{A}\, dv = \int_{x=0}^{3} \int_{y=0}^{\frac{3-x}{2}} \int_{z=0}^{\frac{3-x-2y}{3}} 6xyz\, dx\, dy\, dz = \frac{27}{160}.$

1.47. (a) $\int_V \underset{\sim}{\nabla} \cdot \underset{\sim}{A}\, dv = \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \int_{z=0}^{l} (\cos\phi)\, r\, dr\, d\phi\, dz = 0$

(b) $\int_V \underset{\sim}{\nabla} \cdot \underset{\sim}{A}\, dv = \int_{r=0}^{a} \int_{\phi=0}^{\pi/2} \int_{z=0}^{l} (\cos\phi)\, r\, dr\, d\phi\, dz = a^2 \frac{l}{2}.$

1.48. (a) $\int_V \underset{\sim}{\nabla} \cdot \underset{\sim}{A}\, dv = \int_{r=0}^{1} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} (4r + 2\cos\theta)\, r^2 \sin\theta\, dr\, d\theta\, d\phi = \frac{2\pi}{3}.$

(b) $\int_V \underset{\sim}{\nabla} \cdot \underset{\sim}{A}\, dv = \int_{r=a}^{b} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} (4r + 2\cos\theta)\, r^2 \sin\theta\, dr\, d\theta\, d\phi = 4\pi(b^4 - a^4).$

1.49. $\underset{\sim}{\nabla} \cdot \underset{\sim}{A} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(xy) = 0.$ Hence, $\oint_S \underset{\sim}{A} \cdot d\underset{\sim}{s} = \int_V \underset{\sim}{\nabla} \cdot \underset{\sim}{A}\, dv = 0$

For each case, we choose a convenient closed surface containing the given surface and then evaluate the required integral as the negative of the sum of the surface integrals over all the surfaces of the closed surface excluding the given surface.

(a) By considering the closed surface formed by the planes $x = 0$, $y = 0$, $z = 0$, and $x + 2y + 3z = 3$, we obtain the required integral as

$$\int \underset{\sim}{A} \cdot d\underset{\sim}{s} = -\int_{y=0}^{3/2} \int_{z=0}^{\frac{3-2y}{3}} [\underset{\sim}{A}]_{x=0} \cdot [-dy\, dz\, \underset{\sim}{i}_x] - \int_{x=0}^{3} \int_{z=0}^{\frac{3-x}{3}} [\underset{\sim}{A}]_{y=0} \cdot [-dx\, dz\, \underset{\sim}{i}_y]$$
$$- \int_{x=0}^{3} \int_{y=0}^{\frac{3-x}{2}} [\underset{\sim}{A}]_{z=0} \cdot [-dx\, dy\, \underset{\sim}{i}_z] = \frac{21}{16}.$$

(b) Using the closed surface formed by $x = 0$, $y = 0$, $z = 0$, $z = 1$, and $r = 1$, we obtain $\int \underset{\sim}{A} \cdot d\underset{\sim}{s} = \frac{1}{2}.$

(c) Using the closed surface formed by $z = 0$ and $r = 1$, we obtain $\int \underset{\sim}{A} \cdot d\underset{\sim}{s} = 0.$

(d) Using the closed surface formed by $\theta = \frac{\pi}{4}$ and $z = 1$, we obtain $\int \underset{\sim}{A} \cdot d\underset{\sim}{s} = 0.$

1.50. Derivation similar to that in spherical coordinates (see Section 1.9).

1.51. Derivation similar to that in spherical coordinates (see Section 1.9).

1.52.

Unit vector	$\mathbf{i}_x$	$\mathbf{i}_y$	$\mathbf{i}_z$	$\mathbf{i}_{rc}$	$\mathbf{i}_\phi$	$\mathbf{i}_{rs}$	$\mathbf{i}_\theta$
curl	0	0	0	0	$(1/r_c)\,\mathbf{i}_z$	0	$(1/r)\,\mathbf{i}_\phi$

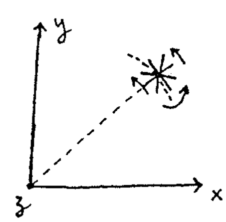
1.53. See page 530 of the text.

1.54. (a) Choosing the z axis as the spin axis, we write $\mathbf{v} = \omega r_c \mathbf{i}_\phi$ where ω is the angular velocity. Paddle wheel does not turn for orientations along the x and y directions. For orientation in the z direction, the paddle wheel turns in the sense of an advancing right hand screw since the paddles for larger values of r_c are hit with greater force than the paddles with smaller values of r_c. Hence, curl has only a z component. In fact, $\nabla \times \mathbf{v} = \nabla \times \omega r_c \mathbf{i}_\phi = 2\omega\, \mathbf{i}_z$.

(b) The position vector field is given by $\mathbf{r} = r_s \mathbf{i}_{rs}$. The paddle wheel does not turn for any orientation since the blades on either side of a position vector line cutting through its axis are hit by equal forces. In fact, $\nabla \times \mathbf{r} = \nabla \times r_s \mathbf{i}_{rs} = 0$.

(c) The field is given by $\mathbf{v} = \dfrac{v_m x}{d}\mathbf{i}_z$. Paddle wheel tuns in the sense of a receding right hand screw when its axis is oriented along the y direction. Paddle wheel does not turn for orientations along the x and z directions. Hence, the curl is entirely in the negative y direction. In fact, $\nabla \times \mathbf{v} = -\dfrac{v_m}{d}\mathbf{i}_y$.

(d) $\mathbf{F} = \mathbf{i}_\phi$. Paddle wheel does not turn for orientations along the x and y directions. For orientation along the z direction, it turns in the sense of an advancing right hand screw since more blades are hit on the far side (see figure) than the near side with the same force density. Hence, the curl has a component



in the z direction. In fact, $\nabla \times \mathbf{F} = \nabla \times \mathbf{i}_\phi = \frac{1}{r}\mathbf{i}_z$.

1.55.

$$\nabla \cdot \nabla \times \mathbf{F} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \equiv 0 \qquad \nabla \times \nabla V = \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{vmatrix} \equiv 0$$

1.56. Any vector for which the divergence is zero can be expressed as the curl of another vector. Any vector for which the curl is zero can be expressed as the gradient of a scalar. $\mathbf{A}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{F}$ can be expressed as curls of some other vectors. $\mathbf{A}$, $\mathbf{C}$, $\mathbf{E}$, and $\mathbf{F}$ can be expressed as gradients of some scalars.

1.57. $\nabla \times \mathbf{A} = -y\,\mathbf{i}_x - z\,\mathbf{i}_y - x\,\mathbf{i}_z$.

$$\oint \mathbf{A} \cdot d\mathbf{l} = \int \nabla \times \mathbf{A} \cdot d\mathbf{S} = \int_{x=0}^{1} \int_{y=0}^{1} (-x)\,dx\,dy + \int_{z=0}^{1} \int_{x=0}^{1-z} z\,dx\,dz$$

$$+ \int_{y=0}^{1} \int_{z=0}^{1-y} y\,dy\,dz = -\frac{2}{3}.$$

1.58. $\nabla \times \mathbf{A} = (2 + 2\sin\phi)\,\mathbf{i}_z$.

(a) $\oint \mathbf{A} \cdot d\mathbf{l} = \int \nabla \times \mathbf{A} \cdot d\mathbf{S} = \int_{r=0}^{1} \int_{\phi=0}^{\pi/2} (2+2\sin\phi)\,r\,dr\,d\phi = 1 + 0.5\,\pi$

(b) $\oint \mathbf{A} \cdot d\mathbf{l} = \int \nabla \times \mathbf{A} \cdot d\mathbf{S} = \int_{r=a}^{b} \int_{\phi=0}^{2\pi} (2+2\sin\phi)\,r\,dr\,d\phi = 2\pi\,(b^2-a^2)$.

1.59. $\nabla \times \mathbf{A} = -\frac{e^{-r}}{r}\,\mathbf{i}_\phi$.

$$\oint \mathbf{A} \cdot d\mathbf{l} = \int \nabla \times \mathbf{A} \cdot d\mathbf{S} = \int_{r=0}^{1} \int_{\theta=0}^{\pi/2} -e^{-r}\,dr\,d\theta = \frac{\pi}{2}\,(e^{-1}-1).$$

1.60. $\nabla \times \mathbf{A} = 0$. Hence $\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{A} \cdot d\mathbf{S} \equiv 0$. For case (a) and (b), we choose a convenient closed path containing the given path and then evaluate the required line integral as the negative of the sum of the line integrals over all the paths of the closed path excluding the given path.

(a) By considering the closed path given by $r = t$, $\phi = \frac{\pi}{2}t$, $z = \sin \pi t$

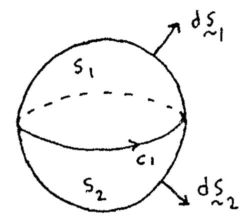
from the origin to $(1, \frac{\pi}{2}, 0)$ and $y=0$ back to the origin, we obtain the required integral as $-\int_{1,\frac{\pi}{2},0}^{0,0,0} [\mathbf{A}]_{y=0} \cdot dy\, \mathbf{i}_y = 0$.

(b) Using the closed path consisting of $x = \sqrt{2}\sin t$, $y = \sqrt{2}\sin t$, $z = \frac{4}{\pi} t$ from $(0,0,0)$ to $(1,1,1)$; $x = 1$, $y = 1$ from $(1,1,1)$ to $(1,1,0)$; and $x = y$ from $(1,1,0)$ to $(0,0,0)$, we obtain $\int_C \mathbf{A} \cdot d\mathbf{l} = 1$.

(c) For any arbitrary path, $\mathbf{A} \cdot d\mathbf{l} = d(xyz)$.

$$\int_{0,0,0}^{22.34,\,5.68,\,-6.93} \mathbf{A} \cdot d\mathbf{l} = [xyz]_{0,0,0}^{22.34,\,5.68,\,-6.93} = -87.93.$$

1.61. For a volume V bounded by the closed surface in the field of $\mathbf{A}$, stokes' theorem yields



$$\int_V \nabla \cdot (\nabla \times \mathbf{A})\, dv = \oint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$

Representing S as the sum of two surfaces S_1 and S_2 (see figure), we write

$$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \int_{S_1} (\nabla \times \mathbf{A}) \cdot d\mathbf{S} + \int_{S_2} (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_{C_1} \mathbf{A} \cdot d\mathbf{l} - \oint_{C_1} \mathbf{A} \cdot d\mathbf{l} = 0$$

Hence, $\int_V \nabla \cdot (\nabla \times \mathbf{A})\, dv = 0$ or, $\nabla \cdot \nabla \times \mathbf{A} = 0$

1.62. $\oint_C \nabla V \cdot d\mathbf{l} = \oint_C dV = 0$ since C is a closed path and hence the lower and upper limits of the integral are the same (provided V is a single valued function). From stokes' theorem, we then have

$$\int_S (\nabla \times \nabla V) \cdot d\mathbf{S} = \oint \nabla V \cdot d\mathbf{l} = 0 \quad \text{or,} \quad \nabla \times \nabla V = 0$$

1.63. For answers, see page 530 of the text.

1.64. $\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\mathbf{i}_x + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\mathbf{i}_y$$

$$+ \frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\mathbf{i}_z - \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) & \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) & \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \end{vmatrix}$$

$$= (\nabla^2 A_x)\mathbf{i}_x + (\nabla^2 A_y)\mathbf{i}_y + (\nabla^2 A_z)\mathbf{i}_z.$$

1.65. Derivation similar to that in cartesian coordinates (See solution for Problem 1.64).

1.66. Derivation similar to that in cartesian coordinates (See solution for Problem 1.64).

1.67. See Sections 1.5, 1.8, 1.9, and 1.10 for expressions for ∇V, $\nabla \cdot \mathbf{J}$, $\nabla \times \mathbf{F}$, and $\nabla^2 V$, respectively, in the different coordinate systems.

1.68. (a) $$\nabla \cdot U\mathbf{A} = \frac{\partial}{\partial x}(UA_x) + \frac{\partial}{\partial y}(UA_y) + \frac{\partial}{\partial z}(UA_z)$$

$$= A_x \frac{\partial U}{\partial x} + A_y \frac{\partial U}{\partial y} + A_z \frac{\partial U}{\partial z} + U\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$$

$$= \mathbf{A} \cdot \nabla U + U \nabla \cdot \mathbf{A}$$

(b)

$$\nabla \times U\mathbf{A} = \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ UA_x & UA_y & UA_z \end{vmatrix} = \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ U\frac{\partial}{\partial x} & U\frac{\partial}{\partial y} & U\frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \nabla U \times \mathbf{A} + U \nabla \times \mathbf{A}.$$

(c)

$$\nabla \cdot \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} B_x & B_y & B_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} - \begin{vmatrix} A_x & A_y & A_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$$

(d)

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (A_y B_z - A_z B_y) & (A_z B_x - A_x B_z) & (A_x B_y - A_y B_x) \end{vmatrix}$$

which upon expansion and simplification gives the desired result.

CHAPTER 2

2.1. For the electric force to counteract the gravitational force, $q\underset{\sim}{E} = -mg\,\underset{\sim}{i}_r$

or, $\underset{\sim}{E} = -\dfrac{mg}{q}\,\underset{\sim}{i}_r$. For an electron, $\underset{\sim}{E} = \dfrac{9.1083\times10^{-31}\times9.8}{1.6021\times10^{-19}} = 55.72\times10^{-12}$ N/C.

2.2. For the charge to follow a circular orbit, the electric field acting on it must exert the required centipetal force. Thus for an orbit of radius r_0,

$$\frac{qE_0}{r_0}\,\underset{\sim}{i}_r = -\frac{mv_0^2}{r_0}\,\underset{\sim}{i}_r \quad \text{or,} \quad E_0 = -\frac{mv_0^2}{q}.$$

2.3. (a) The equations of motion for the test charge are $m\dfrac{dv_x}{dt} = 0$ and $m\dfrac{dv_y}{dt} = qE_0$. Solving these equations and using the initial conditions $v_x = v_0$ and $v_y = 0$ for $t=0$ and $x=0$ and $y=0$ for $t=0$, we get $x = v_0 t$ and $y = \dfrac{q}{2m}E_0 t^2$, or, $y = \dfrac{qE_0}{2mv_0^2}x^2$ which is the equation of a parabola.

(b) The test charge spends a time L/v_0 in the field region. Hence,

$y_L = [y]_{t=L/v_0} = \dfrac{qE_0L^2}{mv_0^2}$. The y component of the velocity is

$\left[\dfrac{dy}{dt}\right]_{t=L/v_0} = \dfrac{qE_0L}{mv_0}\,\underset{\sim}{i}_y$. Thus $\underset{\sim}{v}_L = v_0\,\underset{\sim}{i}_x + \dfrac{qE_0L}{mv_0}\,\underset{\sim}{i}_y$.

(c) Once the charge emerges from the field region, it follows a straight line path along the direction of $\underset{\sim}{v}_L$. Since the time taken by the charge to reach the $x = L+d$ plane from the $x=L$ plane is $\dfrac{d}{v_0}$, we obtain

$$y_d = y_L + \left(\frac{qE_0L}{mv_0}\right)\left(\frac{d}{v_0}\right) = \frac{qE_0L}{mv_0^2}\left(\frac{L}{2}+d\right).$$

2.4. For the test charge to experience no force, the sum of the components of forces acting on it along $y=x$ must be zero since the sum of the components normal to $y=x$ is zero irrespective of the value of k. Thus

$$\frac{Qq}{8\pi\epsilon_0x^2} + \frac{2kQq[(x-1)x+x^2]}{4\pi\epsilon_0[x^2+(x-1)^2]^{3/2}\sqrt{2}\,x} = 0$$

or, $k = -\dfrac{(2x^2+2x+1)^{1/2}}{2\sqrt{2}\,x^2(2x-1)}$. For $x=1$, $k = -\dfrac{1}{2\sqrt{2}} = -0.3535$; for $x = \dfrac{1}{4}$, $k = 5.59$.

2.5. For the charges to be in equilibrium, the resultant of the electric force and the gravitational force acting on each charge must be along the line of the string to that charge. Thus, for an equilateral tetrahedron,

$$\frac{Q^2}{4\pi\epsilon_0 L^2}\Big/ mg = \frac{L/\sqrt{3}}{\sqrt{L^2-\frac{L^2}{3}}} \qquad \text{or,} \qquad \frac{Q^2\sqrt{6}}{4\pi\epsilon_0 L^2 mg} = 1.$$

2.6. (a) The force experienced by each charge is the vector sum of all the forces acting on it due to all the other charges. It is equal to $\frac{3.29}{4\pi\epsilon_0}$ newtons and directed away from the corner opposite to that charge.

(b) The electric field intensity at the point (2,2,2) is the superposition of the electric field intensities at that point due to all the charges. It is equal to $\frac{0.2}{\pi\epsilon_0}(\underset{\sim}{i}_x + \underset{\sim}{i}_y + \underset{\sim}{i}_z)$.

(c) $\frac{1}{4\pi\epsilon_0}(-0.35\,\underset{\sim}{i}_x - 0.35\,\underset{\sim}{i}_y + 1.936\,\underset{\sim}{i}_z)$.

2.7. (a)
$$E_z = \frac{Q}{4\pi\epsilon_0(z-d)^2} - \frac{2Q}{4\pi\epsilon_0 z^2} + \frac{Q}{4\pi\epsilon_0(z+d)^2}$$

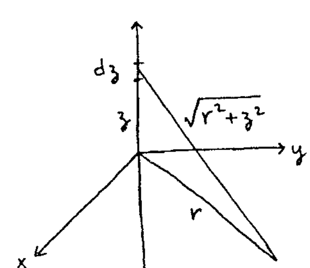
$$= \frac{Q}{4\pi\epsilon_0 z^2}\left[1+\frac{2d}{z}+3\left(\frac{d}{z}\right)^2+\cdots\right] - \frac{2Q}{4\pi\epsilon_0 z^2} + \frac{Q}{4\pi\epsilon_0 z^2}\left[1-\frac{2d}{z}+3\left(\frac{z}{d}\right)^2-\cdots\right]$$

$$= \frac{Q}{4\pi\epsilon_0 z^2}\left[6\left(\frac{d}{z}\right)^2+\cdots\right]$$

For $z \gg d$, $E_z \approx \frac{6Qd^2}{4\pi\epsilon_0 z^4}$.

(b) For $r \gg d$, $E_r \approx -\frac{3Qd^2}{4\pi\epsilon_0 r^4}$.

2.8.
$$d\underset{\sim}{E} = \frac{\rho_L\,dz}{4\pi\epsilon_0(r^2+z^2)}\left[\frac{r}{\sqrt{r^2+z^2}}\underset{\sim}{i}_r - \frac{z}{\sqrt{r^2+z^2}}\underset{\sim}{i}_z\right]$$



The given integrals for E_r, E_ϕ, and E_z follow from this expression.

(a) $E_r = \frac{\rho_{L0}}{2\pi\epsilon_0 r}$, $E_\phi = 0$, $E_z = 0$

(b) $E_r = \frac{\rho_{L0}}{2\pi\epsilon_0 r}\frac{z_0}{\sqrt{r^2+z_0^2}}$, $E_\phi = 0$, $E_z = 0$. For $z_0 \to \infty$, $E_r \to \frac{\rho_{L0}}{2\pi\epsilon_0 r}$.

(c) $E_r = \frac{1}{2\pi\epsilon_0}\left(1-\frac{r}{\sqrt{r^2+z_0^2}}\right)$, $E_\phi = 0$, $E_z = 0$. For large r, $E_r \to \frac{z_0^2}{4\pi\epsilon_0 r^2}$

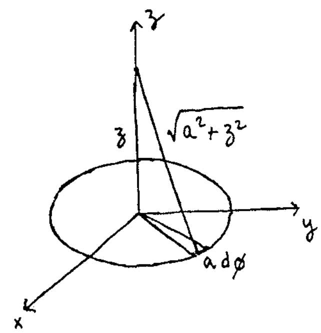
which is equal to the total charge divided by $4\pi\epsilon_0 r^2$.

(d) $E_r = 0,\ E_\phi = 0,\ E_z = \dfrac{1}{2\pi\epsilon_0}\left[\dfrac{z_0}{\sqrt{r^2+z^2}} - \ln\dfrac{z_0+\sqrt{r^2+z_0^2}}{r}\right].$

For large r, $E_z = \dfrac{z_0}{r}\left(1-\dfrac{z_0^2}{2r^2}+\cdots\right) - \ln\left(1+\dfrac{z_0}{r}+\dfrac{z_0^2}{2r^2}+\cdots\right)$

$$\rightarrow -\frac{z_0^3}{3r^3} = -\frac{\text{dipole moment}}{4\pi\epsilon_0 r^3}$$

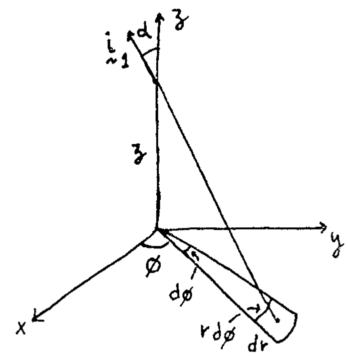
2.9. $d\underset{\sim}{E} = \dfrac{\rho_L a\, d\phi}{4\pi\epsilon_0 (a^2+z^2)}\left[\dfrac{a}{\sqrt{a^2+z^2}}\underset{\sim}{i}_r + \dfrac{z}{\sqrt{a^2+z^2}}\underset{\sim}{i}_z\right]$



$dE_x = -dE_r\cos\phi$, $dE_y = -dE_r\sin\phi$

The given integrals for E_x, E_y, and E_z follow from these expressions. For answers, see page 530 of the text.

2.10. $d\underset{\sim}{E} = \dfrac{\rho_s r\, dr\, d\phi}{4\pi\epsilon_0 (r^2+z^2)}\underset{\sim}{i}_1$



$dE_z = dE\cos\alpha = \dfrac{\rho_s z r\, dr\, d\phi}{4\pi\epsilon_0 (r^2+z^2)^{3/2}}$

$dE_r = dE\sin\alpha = \dfrac{\rho_s r^2\, dr\, d\phi}{4\pi\epsilon_0 (r^2+z^2)^{3/2}}$

$dE_x = -dE_r\cos\phi$, $dE_y = -dE_r\sin\phi$

The given integrals for E_x, E_y, and E_z follow from these expressions.

(a) $E_x = E_y = 0,\ E_z = \dfrac{\rho_{s0} z}{2\epsilon_0 |z|}$

(b) $E_x = E_y = 0,\ E_z = \dfrac{\rho_{s0}}{2\epsilon_0}\left(\dfrac{z}{|z|} - \dfrac{z}{\sqrt{r_0^2+z^2}}\right)$

For $|z| \gg r_0$, $E_z \rightarrow \dfrac{\text{total charge}}{4\pi\epsilon_0 z^2}$; for $r_0 \rightarrow \infty$, $E_z \rightarrow \dfrac{\rho_{s0}}{2\epsilon_0}$.

(c) $E_x = E_y = 0$, $E_z = \dfrac{\rho_{s0}}{2\epsilon_0}\dfrac{z}{\sqrt{r_0^2+z^2}}$

For $z \rightarrow 0$, $E_z \rightarrow 0$; for $r_0 \rightarrow 0$, $E_z \rightarrow \dfrac{\rho_{s0}}{2\epsilon_0}$.

(d) $E_x = -\dfrac{\rho_{s0}}{4\epsilon_0 |z|}$, $E_y = E_z = 0$

(e) $E_x = E_z = 0$, $E_y = -\dfrac{\rho_{s0}}{4\epsilon_0 |z|}$

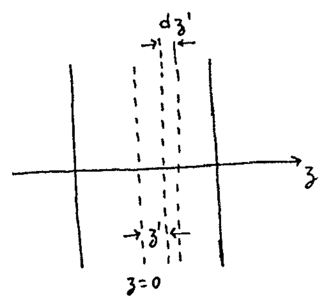
2.11. We consider the spherical surface charge to be made up of a series of ring charges, $r_s = a$ and θ = constant. Since the charge density is independent of ϕ, the electric field intensity at $(0, 0, z)$ due to each ring charge is entirely z directed as shown in Problem 2.9(a). Using the result of Problem 2.9(a), we obtain, for a ring charge corresponding to an arbitrary value of θ,

$$dE_z = \frac{(2\pi a \sin\theta)(\rho_s\, a\, d\theta)(z - a\cos\theta)}{4\pi\epsilon_0 [(a\sin\theta)^2 + (z - a\cos\theta)^2]^{3/2}}$$

which gives the required result for E_z.

For answers to (a) and (b), see page 530 of the text.

2.12. We divide the volume charge into a series of sheet charges of infinitesimal thickness dz'. Let us consider one such sheet charge located at $z = z'$. We then have



$$d\mathbf{E} = \begin{cases} \dfrac{\rho\, dz'}{2\epsilon_0}\,\mathbf{i}_z & \text{for } z > z' \\ -\dfrac{\rho\, dz'}{2\epsilon_0}\,\mathbf{i}_z & \text{for } z < z' \end{cases}$$

which gives

$$E_z = \begin{cases} \displaystyle\int_{z'=-a}^{a} \frac{\rho\, dz'}{2\epsilon_0} = \frac{1}{2\epsilon_0}\int_{z=-a}^{a} \rho\, dz & \text{for } z > a \\ \displaystyle\int_{z'=-a}^{z} \frac{\rho\, dz'}{2\epsilon_0} - \int_{z'=z}^{a} \frac{\rho\, dz'}{2\epsilon_0} = \frac{1}{2\epsilon_0}\left(\int_{z=-a}^{z} \rho\, dz - \int_{z=z}^{a} \rho\, dz\right) & \text{for } -a < z < a \\ \displaystyle -\int_{z'=-a}^{a} \frac{\rho\, dz'}{2\epsilon_0} = -\frac{1}{2\epsilon_0}\int_{z=-a}^{a} \rho\, dz & \text{for } z < -a \end{cases}$$

(a) $\dfrac{\rho_0 z}{\epsilon_0}$ for $|z| < a$, $\dfrac{\rho_0 a |z|}{\epsilon_0 z}$ for $|z| > a$

(b) $\dfrac{\rho_0}{\epsilon_0}(|z| - a)$ for $|z| < a$, 0 for $|z| > a$

(c) $\dfrac{z^3}{2\epsilon_0 |z|}$ for $|z| < a$, $\dfrac{a^2 |z|}{2\epsilon_0 z}$ for $|z| > a$

(d) $\dfrac{z^2 - a^2}{2\epsilon_0}$ for $|z| < a$, 0 for $|z| > a$

2.13. Derivation follows along lines similar to Example 2-6. For answer, see page 530 of the text. Result can also be obtained by directly considering the cylindrical charge as a superposition of infinitely long line charges.

2.14. (a) From Example 2-6, the electric field intensity at any radius $r \le a$ is equal to $\dfrac{Qr}{4\pi\epsilon_0 a^3}\,\underset{\sim}{i}_r$. Hence the equation of motion of the charge is

$$m\frac{d^2x}{dt^2} + \frac{Q|q|x}{4\pi\epsilon_0 a^3} = 0$$, where x is the distance from the center.

(b) $x = a\cos\sqrt{\dfrac{Q|q|}{4\pi\epsilon_0 m a^3}}\,t$, $v = -a\sqrt{\dfrac{Q|q|}{4\pi\epsilon_0 m a^3}}\sin\sqrt{\dfrac{Q|q|}{4\pi\epsilon_0 m a^3}}\,t$.

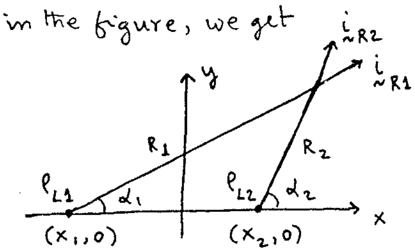
(c) $\dfrac{1}{2\pi}\sqrt{\dfrac{Q|q|}{4\pi\epsilon_0 m a^3}}$.

2.15. Derivation follows along lines similar to Example 2-3. For answers, see page 530 of the text.

2.16. Setting up the coordinate system as shown in the figure, we get

$$E_x = \frac{\rho_{L1}}{2\pi\epsilon_0}\frac{x-x_1}{R_1^2} + \frac{\rho_{L2}}{2\pi\epsilon_0}\frac{x-x_2}{R_2^2}$$

$$E_y = \frac{\rho_{L1}}{2\pi\epsilon_0}\frac{y}{R_1^2} + \frac{\rho_{L2}}{2\pi\epsilon_0}\frac{y}{R_2^2}$$



Substituting for E_x and E_y in $\dfrac{dy}{E_y} = \dfrac{dx}{E_x}$ and cross multiplying, we obtain

$$\rho_{L1}\frac{(x-x_1)\,dy - y\,dx}{(x-x_1)^2+y^2} + \rho_{L2}\frac{(x-x_2)\,dy - y\,dx}{(x-x_2)^2+y^2} = 0$$

$$\rho_{L1}\frac{d\left(\frac{y}{x-x_1}\right)}{1+\left(\frac{y}{x-x_1}\right)^2} + \rho_{L2}\frac{d\left(\frac{y}{x-x_2}\right)}{1+\left(\frac{y}{x-x_2}\right)^2} = 0$$

$$\rho_{L1}\frac{d(\tan\alpha_1)}{\sec^2\alpha_1} + \rho_{L2}\frac{d(\tan\alpha_2)}{\sec^2\alpha_2} = 0$$

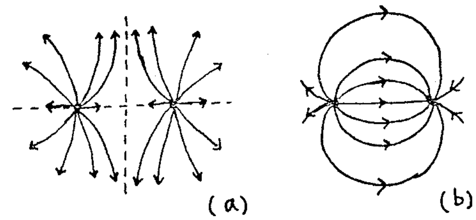
$$\rho_{L1}\,d\alpha_1 + \rho_{L2}\,d\alpha_2 = 0$$

$$\rho_{L1}\alpha_1 + \rho_{L2}\alpha_2 = \text{Constant}.$$

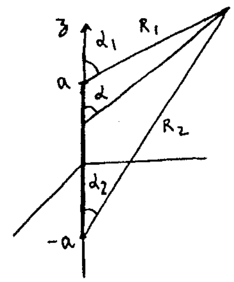
The result can be generalized for any number of line charges.

(a) $\alpha_1 + \alpha_2 = $ constant

(b) $\alpha_1 - \alpha_2 = $ constant



2.17. Locating the line charge along the z-axis between $z = -a$ and $z = +a$ and considering an infinitesimal element as shown in the figure, we obtain



$$d\underset{\sim}{E} = \frac{\rho_{L0}\, dz' \,(\cos\alpha \, \underset{\sim}{i}_z + \sin\alpha \, \underset{\sim}{i}_r)}{4\pi\epsilon_0 \,[r^2 + (z-z')^2]}$$

which gives the electric field intensity components due to the entire line charge as

$$E_z = \frac{\rho_{L0}}{4\pi\epsilon_0 r}(\sin\alpha_1 - \sin\alpha_2) = \frac{\rho_{L0}}{4\pi\epsilon_0 r}\left(\frac{r}{R_1} - \frac{r}{R_2}\right)$$

$$E_r = \frac{\rho_{L0}}{4\pi\epsilon_0 r}(\cos\alpha_2 - \cos\alpha_1) = \frac{\rho_{L0}}{4\pi\epsilon_0 r}\left(\frac{z+a}{R_1} - \frac{z-a}{R_2}\right)$$

Substituting for E_r and E_z in $\frac{dr}{E_r} = \frac{dz}{E_z}$ and crossmultiplying, we obtain the equation for the direction lines as

$$\frac{(z+a)\,dz + r\,dr}{R_2} = \frac{(z-a)\,dz + r\,dr}{R_1}$$

$$dR_2 = dR_1$$

$$R_2 - R_1 = \text{Constant.}$$

This is the equation of hyperbolas with their focii at the ends of the line charge.

2.18. We can represent a surface S which does not enclose the point charge as the superposition of two surfaces S_1 and S_2 both of which enclose the point charge but with the normal to one of the surfaces (say S_1) outward and the normal to the other surface (S_2) inward. Then,

$$\oint_S \underset{\sim}{E}\cdot d\underset{\sim}{S} = \oint_{S_1} \underset{\sim}{E}\cdot d\underset{\sim}{S} + \oint_{S_2} \underset{\sim}{E}\cdot d\underset{\sim}{S} = \frac{Q}{\epsilon_0} - \frac{Q}{\epsilon_0} = 0.$$

2.19. All answers follow from symmetry considerations. See page 530 of the text for answers

2.20. $$\int \underset{\sim}{E}\cdot d\underset{\sim}{S} = \int_{x=0}^{1}\int_{z=0}^{1} \frac{\rho_{L0}}{2\pi\epsilon_0 r}\,\underset{\sim}{i}_r \cdot(\underset{\sim}{i}_x + \underset{\sim}{i}_y)\, dx\, dz = \frac{\rho_{L0}}{4\epsilon_0}.$$

The total flux emanating from the line charge between $z=0$ and $z=1$ is ρ_{L0}/ϵ_0. From symmetry considerations, one fourth of this flux goes into the first quadrant, which checks with the answer obtained by evaluating $\int \mathbf{E} \cdot d\mathbf{S}$.

2.21. $$\int \mathbf{E} \cdot d\mathbf{S} = \int_{x=0}^{1} \int_{z=0}^{\infty} \frac{Q}{4\pi\epsilon_0 r_s^2} \mathbf{i}_{rs} \cdot (\mathbf{i}_x + \mathbf{i}_y)\, dx\, dz = \frac{Q}{8\epsilon_0}.$$

The total flux emanating from the point charge is $\frac{Q}{\epsilon_0}$. From symmetry considerations, 1/8 of the total flux goes into the first octant, which checks with the answer obtained by evaluating $\int \mathbf{E} \cdot d\mathbf{S}$

2.22. From considerations of symmetry and geometry, the electric field fluxes cutting the given surface are as follows: (a) $\frac{1}{2\epsilon_0}$, (b) $\frac{\sqrt{1-0.36}}{\epsilon_0} = \frac{0.8}{\epsilon_0}$

(c) $\frac{\pi(1-0.25)}{\epsilon_0} = \frac{0.375\pi}{\epsilon_0}$. Hence the total flux $= \frac{2.478}{\epsilon_0}$.

2.23. The solution for each part consists of choosing an appropriate Gaussian surface from considerations of symmetry and then applying Gauss' law. For answers, see page 531 of the text. The Gaussian surfaces are as follows:

(a) Rectangular box with one surface located in the $z=0$ plane

(b) Rectangular box with one surface located in the $z=a$ (or $z=-a$) plane

(c) Same as for (a)

(d) Same as for (b)

(e) Same as for (a)

2.24. Gaussian surfaces are cylinders having the z-axis as their axes.

(a) $\frac{\rho_0 r}{2\epsilon_0} \mathbf{i}_r$ for $r<a$, $\frac{\rho_0 a^2}{2\epsilon_0 r} \mathbf{i}_r$ for $r>a$

(b) 0 for $r<a$, $\frac{\rho_0}{2\epsilon_0 r}(r^2-a^2)\, \mathbf{i}_r$ for $a<r<b$, $\frac{\rho_0}{2\epsilon_0 r}(b^2-a^2)\, \mathbf{i}_r$ for $r>b$

(c) $\frac{\rho_0 r^2}{3\epsilon_0 a} \mathbf{i}_r$ for $r<a$, $\frac{\rho_0 a^2}{3\epsilon_0 r} \mathbf{i}_r$ for $r>a$

2.25. Gaussian surfaces are spheres having centers at the origin. See page 531 of the text for answers.

2.26. (a) $-\frac{\rho_{s0}}{\epsilon_0} \mathbf{i}_z$ for $|z|<a$, 0 for $|z|>a$

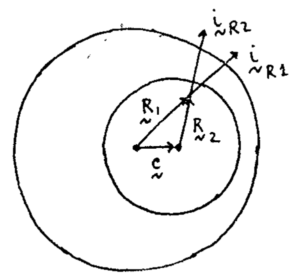
(b) 0 for $r<a$, $\frac{\rho_{s0}\, a}{\epsilon_0 r} \mathbf{i}_r$ for $r>a$

(c) 0 for $r<a$, $\frac{\rho_{s0}a}{\epsilon_0 r}\,\underset{\sim}{i}_r$ for $a<r<b$, 0 for $r>b$

(d) 0 for $r<a$, $\frac{\rho_{s0}a^2}{\epsilon_0 r^2}\,\underset{\sim}{i}_r$ for $r>a$

(e) 0 for $r<a$, $\frac{\rho_{s0}a^2}{\epsilon_0 r^2}\,\underset{\sim}{i}_r$ for $a<r<b$, 0 for $r>b$

2.27. We make use of superposition to solve this problem by considering the given charge distribution as the sum of two uniformly distributed cylindrical charges, one of radius a and the other of radius b, and such that the total charge in the hole is zero. Thus, we obtain the required electric field intensity as



$$\underset{\sim}{E} = \frac{\rho_0 R_1}{2\epsilon_0}\,\underset{\sim}{i}_{R1} - \frac{\rho_0 R_2}{2\epsilon_0}\,\underset{\sim}{i}_{R2}$$

$$= \frac{\rho_0}{2\epsilon_0}(\underset{\sim}{R}_1 - \underset{\sim}{R}_2) = \frac{\rho_0}{2\epsilon_0}\underset{\sim}{c}$$

Thus the electric field inside the hole is uniform having magnitude $\frac{\rho_0 c}{2\epsilon_0}$ and directed parallel to the line joining the two axes.

2.28. All fields have only z components. Hence Gauss' law in differential form simplifies to $\frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon_0}$. Verification consists of substituting E_z obtained in Problem 2.23 to check if ρ agrees with that given in Problem 2.23.

2.29. $\frac{1}{r}\frac{\partial}{\partial r}(rE_r) = \frac{\rho}{\epsilon_0}$. Procedure similar to Problem 2.28.

2.30. $\frac{1}{r^2}\frac{\partial}{\partial r}(r^2E_r) = \frac{\rho}{\epsilon_0}$. Procedure similar to Problem 2.28.

2.31. (a) $\rho = \epsilon_0 \underset{\sim}{\nabla}\cdot\underset{\sim}{E} = \epsilon_0 \frac{\partial E_z}{\partial z} = \frac{2}{3}\rho_{s0}\,\delta(z) + \frac{4}{3}\rho_{s0}\,\delta(z-a)$.

(b) $\rho = \epsilon_0 \underset{\sim}{\nabla}\cdot\underset{\sim}{E} = \epsilon_0 \frac{1}{r}\frac{\partial}{\partial r}(rE_r) = \frac{e^{-r}}{r}$.

(c) $\rho = \epsilon_0 \underset{\sim}{\nabla}\cdot\underset{\sim}{E} = \epsilon_0 \frac{1}{r^2}\frac{\partial}{\partial r}(r^2E_r) = \frac{Q}{4\pi a^2}\,\delta(r-a) - \frac{Q}{4\pi b^2}\,\delta(r-b)$.

2.32. We start with a volume charge of density ρ independent of r and lying between the spherical surfaces $r = r_0 - \Delta r$ and $r = r_0 + \Delta r$. The total charge per unit surface area between $r_0 - \Delta r$ and $r_0 + \Delta r$ is equal to

$2\rho\Delta r$. Let this quantity be ρ_S. If we now let Δr tend to zero increasing ρ such that ρ_S remains constant, we obtain a surface charge of density ρ_S C/m^2. The quantity ρ becomes a delta function $\rho_S\,\delta(r-r_0)$ since $\int_{r=r_0^-}^{r_0^+} \rho_S\,\delta(r-r_0)\,dr = \rho_S$. Thus, we have $\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho_S\,\delta(r-r_0)$.

2.33. We start with a volume charge lying between the four surfaces $r = r_0 \pm \Delta r$ and $\phi = \phi_0 \pm \Delta\phi$ and of uniform density ρ C/m^3. The total charge per unit length in the z direction is equal to $\rho_0\,(2\Delta\phi)\,\dfrac{(r_0+\Delta r)^2-(r_0-\Delta r)^2}{2}$. Let this quantity be ρ_{L0}. If we now let Δr and $\Delta\phi$ tend to zero increasing ρ_0 such that ρ_{L0} remains constant, we obtain a line charge of density ρ_{L0} C/m. The quantity ρ becomes a delta function $\dfrac{\rho_{L0}\,\delta(r-r_0)\,\delta(\phi-\phi_0)}{r_0}$ since

$$\int_{r=r_0^-}^{r_0^+}\int_{\phi=\phi_0^-}^{\phi_0^+} \frac{\rho_{L0}\,\delta(r-r_0)\,\delta(\phi-\phi_0)}{r_0}\, r\,dr\,d\phi = \rho_{L0}.$$ Thus, we have

$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho_{L0}\,\frac{\delta(r-r_0)\,\delta(\phi-\phi_0)}{r_0}.$$

2.34. Solution follows in a manner similar to those of Problems 2.32. and 2.33. by starting with a volume charge lying between the six surfaces $r = r_0 \pm \Delta r$, $\theta = \theta_0 \pm \Delta\theta$, and $\phi = \phi_0 \pm \Delta\phi$.

2.35. $$\text{Work} = \int_{1,0,-22.7}^{0.5,\frac{\pi}{2},43.8} \mathbf{E}\cdot d\mathbf{l} = \int_{1,0,-22.7}^{0.5,\frac{\pi}{2},43.8} \left(\frac{\cos\phi}{r^2}\mathbf{i}_r + \frac{\sin\phi}{r^2}\mathbf{i}_\phi\right)\cdot\left(dr\,\mathbf{i}_r + r\,d\phi\,\mathbf{i}_\phi + dz\,\mathbf{i}_z\right)$$

$$= \left[-\frac{\cos\phi}{r}\right]_{1,0,-22.7}^{0.5,\frac{\pi}{2},43.8} = 1.$$

The work is done by the field.

2.36. $$V_1 - V_x = \int_1^x \mathbf{E}\cdot d\mathbf{l}$$

$$= \begin{cases} \displaystyle\int_1^x 2x\,dx = x^2-1 & \text{for } 0<x<1 \\ \displaystyle\int_1^0 2x\,dx + \int_0^x 0\,dx = -1 & \text{for } -\infty<x<0 \\ \displaystyle\int_1^x \frac{2}{x^2}\,dx = 2-\frac{2}{x} & \text{for } 1<x<\infty \end{cases}$$

2.37. The normal vectors to the equipotential surfaces are given by

$$\nabla(r^2 \sec\theta) = 2r\sec\theta\, \mathbf{i}_r + r\sec\theta\tan\theta\, \mathbf{i}_\theta .$$

Hence, the direction lines are given by

$$\frac{dr}{2r\sec\theta} = \frac{r\,d\theta}{r\sec\theta\tan\theta} = \frac{r\sin\theta\, d\phi}{0}$$

$$\frac{dr}{r} = 2\cot\theta\, d\theta, \quad \text{and} \quad d\phi = 0$$

$$r\operatorname{cosec}^2\theta = \text{constant}, \quad \text{and} \quad \phi = \text{constant}.$$

2.38. $$V = \frac{Q}{4\pi\epsilon_0\sqrt{r^2+d^2-2rd\cos\theta}} - \frac{2Q}{4\pi\epsilon_0 r} + \frac{Q}{4\pi\epsilon_0\sqrt{r^2+d^2+2rd\cos\theta}}$$

$$= \frac{Q}{4\pi\epsilon_0 r}\left[\left(1+\frac{d^2}{r^2}-\frac{2d}{r}\cos\theta\right)^{-1/2} - 2 + \left(1+\frac{d^2}{r^2}+\frac{2d}{r}\cos\theta\right)^{-1/2}\right]$$

$$= \frac{Q}{4\pi\epsilon_0 r}\left[1-\frac{1}{2}\left(\frac{d^2}{r^2}-\frac{2d}{r}\cos\theta\right)+\frac{3}{8}\left(\frac{d^2}{r^2}-\frac{2d}{r}\cos\theta\right)^2 - \cdots\right.$$

$$\left. -2+1-\frac{1}{2}\left(\frac{d^2}{r^2}+\frac{2d}{r}\cos\theta\right)+\frac{3}{8}\left(\frac{d^2}{r^2}+\frac{2d}{r}\cos\theta\right)^2-\cdots\right]$$

$$\approx \frac{Qd^2}{4\pi\epsilon_0 r^3}(3\cos^2\theta - 1).$$

2.39. $$V = \frac{Q}{4\pi\epsilon_0\sqrt{x^2+y^2+z^2}} - \frac{Q}{4\pi\epsilon_0\sqrt{(x-\Delta x)^2+y^2+z^2}}$$

$$- \frac{Q}{4\pi\epsilon_0\sqrt{x^2+y^2+(z-\Delta z)^2}} + \frac{Q}{4\pi\epsilon_0\sqrt{(x-\Delta x)^2+y^2+(z-\Delta z)^2}}$$

$$= \frac{Q}{4\pi\epsilon_0 r}\left\{1 - \left[1-\frac{2\Delta x}{r}\sin\theta\cos\phi + \left(\frac{\Delta x}{r}\right)^2\right]^{-1/2} - \left[1-\frac{2\Delta z}{r}\cos\theta+\left(\frac{\Delta z}{r}\right)^2\right]^{-1/2}\right.$$

$$\left. - \left[1-\frac{2\Delta x}{r}\sin\theta\cos\phi - \frac{2\Delta z}{r}\cos\theta + \left(\frac{\Delta x}{r}\right)^2 + \left(\frac{\Delta z}{r}\right)^2\right]^{-1/2}\right\}$$

$$\approx \frac{3Q\,\Delta x\,\Delta z}{4\pi\epsilon_0 r^3}\sin\theta\cos\theta\cos\phi .$$

2.40. $\Sigma Q_j = 0$

$$\Sigma Q_j \mathbf{r}_j' = Qd\,[(\mathbf{i}_x+\mathbf{i}_y+\mathbf{i}_z) - (\mathbf{i}_x+\mathbf{i}_y) - (\mathbf{i}_y+\mathbf{i}_z) - (\mathbf{i}_z+\mathbf{i}_x) + (\mathbf{i}_x+\mathbf{i}_y+\mathbf{i}_z)] = 0$$

$$\Sigma Q_j\{3(\mathbf{r}_j'\cdot\mathbf{r})^2 - r^2 r_j'^2\}$$

$$= Qd^2r^2\{[3(\mathbf{i}_x\cdot\mathbf{i}_r)^2-1] + [3(\mathbf{i}_y\cdot\mathbf{i}_r)^2-1] + [3(\mathbf{i}_z\cdot\mathbf{i}_r)^2-1]$$

$$-3[(\mathbf{i}_x+\mathbf{i}_y)\cdot\mathbf{i}_r]^2 + 2 - 3[(\mathbf{i}_y+\mathbf{i}_z)\cdot\mathbf{i}_r]^2 + 2$$

$$-3[(\mathbf{i}_z+\mathbf{i}_x)\cdot\mathbf{i}_r]^2 + 2 + 3[(\mathbf{i}_x+\mathbf{i}_y+\mathbf{i}_z)\cdot\mathbf{i}_r]^2 - 3\}$$

$$= 0.$$

Hence, we have to consider the next higher order term, which is the "octupole" term. From the binomial expansion given by Eq. (2-112), this term can be shown to be $\dfrac{1}{24\pi\epsilon_0 r^7}\sum_j Q_j\,[15(\underset{\sim}{r}_j'\cdot\underset{\sim}{r})^3 - 9r_j'^2 r^2(\underset{\sim}{r}_j'\cdot\underset{\sim}{r})]$

which upon substitution for Q_j and $\underset{\sim}{r}_j'$ gives

$$V = \frac{15Qd^3}{4\pi\epsilon_0 r^4}(\sin\theta\cos\phi)(\sin\theta\sin\phi)\cos\theta .$$

Alternatively, the solution can be obtained by applying the result of Problem 2.39. to the individual quadrupoles making up the octupole of this problem.

2.41. (a) $\Sigma Q_j = Q - \frac{Q}{2} + \frac{Q}{4} - \frac{Q}{8} = \frac{5Q}{8}$.

$$\Sigma Q_j \underset{\sim}{r}_j' = Q(\underset{\sim}{i}_x + \underset{\sim}{i}_z) - \frac{Q}{2}(-\underset{\sim}{i}_x + \underset{\sim}{i}_z) + \frac{Q}{4}(-\underset{\sim}{i}_x - \underset{\sim}{i}_z) - \frac{Q}{8}(\underset{\sim}{i}_x - \underset{\sim}{i}_z)$$

$$= Q\left(\frac{9}{8}\underset{\sim}{i}_x + \frac{3}{8}\underset{\sim}{i}_z\right)$$

$$V = \frac{\Sigma Q_j}{4\pi\epsilon_0 r} + \frac{\Sigma Q_j \underset{\sim}{r}_j'\cdot\underset{\sim}{r}}{4\pi\epsilon_0 r^3}.$$

(b) $\Sigma Q_j = -3$, $\Sigma Q_j \underset{\sim}{r}_j' = -4(\underset{\sim}{i}_x + \underset{\sim}{i}_y + \underset{\sim}{i}_z)$, $V = \dfrac{\Sigma Q_j}{4\pi\epsilon_0 r} + \dfrac{\Sigma Q_j \underset{\sim}{r}_j'\cdot\underset{\sim}{r}}{4\pi\epsilon_0 r^3}$

(c) $\Sigma Q_j = 0$, $\Sigma Q_j \underset{\sim}{r}_j' = 2Q(\underset{\sim}{i}_x + \underset{\sim}{i}_z)$

$$\Sigma Q_j[3(\underset{\sim}{r}_j'\cdot\underset{\sim}{r})^2 - r^2 r_j'^2] = Qr^2(6\sin^2\theta\cos^2\phi + 6\cos^2\theta + 6\sin\theta\cos\theta\cos\phi - 4).$$

$$V = \frac{\Sigma Q_j}{4\pi\epsilon_0 r} + \frac{\Sigma Q_j \underset{\sim}{r}_j'\cdot\underset{\sim}{r}}{4\pi\epsilon_0 r^3} + \frac{\Sigma Q_j[3(\underset{\sim}{r}_j'\cdot\underset{\sim}{r})^2 - r^2 r_j'^2]}{8\pi\epsilon_0 r^5}$$

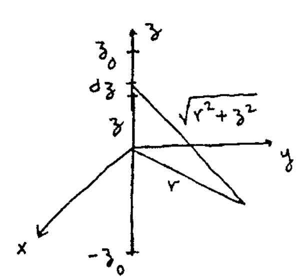
See page 531 of the text for expressions for V.

2.42. Dipole moment about an arbitrary point (x, y, z)

$$= -3Q\underset{\sim}{r} + Q(\underset{\sim}{i}_x - \underset{\sim}{r}) + Q(\underset{\sim}{i}_z - \underset{\sim}{r}) + Q(\underset{\sim}{i}_x + \underset{\sim}{i}_z - \underset{\sim}{r}) = 2Q(\underset{\sim}{i}_x + \underset{\sim}{i}_z)$$

which is the same as that obtained in Problem 2.41 (c).

2.43. $dV = \dfrac{\rho_L\,dz}{4\pi\epsilon_0\sqrt{r^2+z^2}}$, $V = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int_{z=-z_0}^{z_0}\frac{\rho_L\,dz}{\sqrt{r^2+z^2}}$

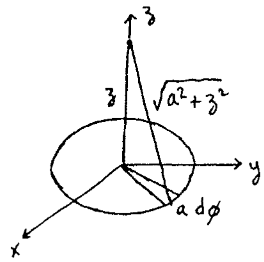


For answers, see page 531 of the text.

Limiting cases: (a) For $r \gg z_0$, the line charge appears like a point charge. For $r \ll z_0$, the expression for V should tend to that for an infinitely long line charge.

(b) For $r \gg z_0$, the line charge appears like a point charge.

2.44. $dV = \dfrac{\rho_L a\, d\phi}{4\pi\epsilon_0\sqrt{a^2+z^2}}$, $V = \dfrac{a}{4\pi\epsilon_0 (a^2+z^2)^{1/2}} \displaystyle\int \rho_L\, d\phi$

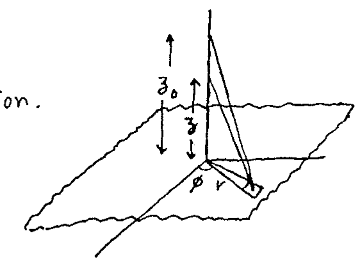


(a) $V = \dfrac{a\rho_{L0}}{2\epsilon_0 (a^2+z^2)^{1/2}}$

For $z \gg a$, $V \approx \dfrac{2\pi a \rho_{L0}}{4\pi\epsilon_0 z} = \dfrac{\text{total charge on the ring}}{4\pi\epsilon_0 z}$.

(b) 0 (c) 0 (d) 0

2.45. $dV = \dfrac{\rho_S\, r\, dr\, d\phi}{4\pi\epsilon_0\sqrt{r^2+z^2}} - \dfrac{\rho_S\, r\, dr\, d\phi}{4\pi\epsilon_0\sqrt{r^2+z_0^2}}$



The given integral for V follows from this expression.

For answers, see page 531 of the text.

Note: In evaluating the integrals, we should write, for example,

$$\int_{r=0}^{\infty} \left[\frac{r\,dr}{(r^2+z^2)^{1/2}} - \frac{r\,dr}{(r^2+z_0^2)^{1/2}} \right] = \left[\left|\sqrt{r_0^2+z_0^2}\right| - \sqrt{\left|\sqrt{r^2+z_0^2}\right|} \right]_{r=0}^{\infty} = |z_0| - |z|$$

since the electric field is directed away from the sheet charge on either side of it and hence the potential must be an even function of z.

Limiting cases: (b) For $z \gg r_0$, the charge appears like a point charge

2.46. The potential due to an infinitesimal amount of charge on an infinitesimal area $(a\,d\theta)(a\sin\theta\, d\phi)$ on the spherical surface is

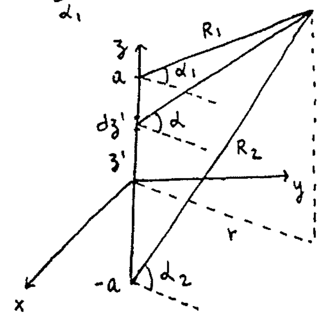
$dV = \dfrac{\rho_S a^2 \sin\theta\, d\theta\, d\phi}{4\pi\epsilon_0\sqrt{a^2+z^2-2az\cos\theta}}$ which gives the required expression for V.

(a) $V = \dfrac{4\pi a^2 \rho_{S0}}{4\pi\epsilon_0 |z|}$ for $|z| > a$, $\dfrac{4\pi a^2\rho_{S0}}{4\pi\epsilon_0 a}$ for $|z| < a$.

(b) $V = \dfrac{\rho_{S0} z}{3\epsilon_0}$ for $|z| < a$, $\dfrac{\rho_{S0} a^3 |z|}{3\epsilon_0 z^3}$ for $|z| > a$.

2.47. $V = \displaystyle\int_{z'=-a}^{a} \frac{\rho_{L0}\, dz'}{4\pi\epsilon_0\sqrt{r^2+(z-z')^2}} = -\frac{\rho_{L0}}{4\pi\epsilon_0}\left[\ln(\sec\alpha+\tan\alpha)\right]_{\alpha_1}^{\alpha_2}$

$= \dfrac{\rho_{L0}}{4\pi\epsilon_0} \ln \dfrac{\sqrt{r^2+(z+a)^2}+(z+a)}{\sqrt{r^2+(z-a)^2}+(z-a)}$.



Equipotential surfaces are given by

$\dfrac{\sqrt{r^2+(z+a)^2}+(z+a)}{\sqrt{r^2+(z-a)^2}+(z-a)}$ = constant, say, c.

$$\sqrt{r^2+(z+a)^2} - c\sqrt{r^2+(z-a)^2} = c(z-a) - (z+a)$$

Squaring both sides, simplifying, and again squaring both sides and simplifying, we obtain

$$(c^2-1)^2 r^2 + 4c(c-1)^2 z^2 - 4c(c+1)^2 a^2 = 0$$

$$\frac{(c-1)^2}{4c}\left(\frac{r}{a}\right)^2 + \frac{(c-1)^2}{(c+1)^2}\left(\frac{z}{a}\right)^2 = 1.$$

This is the equation of an ellipse having semimajor axis $\frac{c+1}{c-1}a$ along the z direction and semiminor axis $\frac{2\sqrt{c}}{c-1}a$ along the r direction.

Distance from center to either focus is $\sqrt{\left(\frac{c+1}{c-1}a\right)^2 - \left(\frac{2\sqrt{c}}{c-1}a\right)^2} = a.$

Thus the equipotential surfaces are ellipsoids with the ends of the line as their focii.

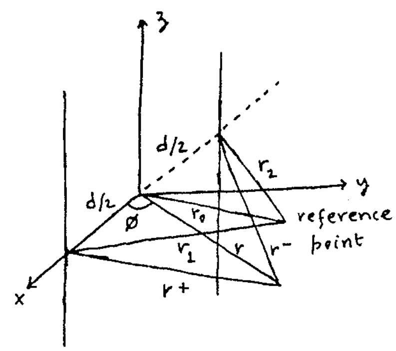
To establish the orthogonality of the equipotential surface to the direction lines of $\underset{\sim}{E}$, we note that

$$\underset{\sim}{\nabla} V = \frac{\rho_{L0}}{4\pi\epsilon_0}\underset{\sim}{\nabla}\left[\ln\frac{\sqrt{r^2+(z+a)^2}+(z+a)}{\sqrt{r^2+(z-a)^2}+(z-a)}\right]$$

$$= \frac{\rho_{L0}}{4\pi\epsilon_0 r}\left[\left(\frac{z-a}{R_1} - \frac{z+a}{R_2}\right)\underset{\sim}{i}_r + \left(\frac{r}{R_2} - \frac{r}{R_1}\right)\underset{\sim}{i}_z\right]$$

which is proportional to $\underset{\sim}{E}$ obtained in Problem 2.17.

2.48. (a) $V = -\frac{\rho_{L0}}{2\pi\epsilon_0}\ln\frac{r^+}{r_1} + \frac{\rho_{L0}}{2\pi\epsilon_0}\ln\frac{r^-}{r_2}$



$$\ln\frac{r^+}{r_1} = \ln\frac{\sqrt{r^2+(d/2)^2 - rd\cos\phi}}{r_1}$$

$$\approx \ln\frac{r}{r_0} - \frac{d}{2r}\cos\phi \quad \text{for } d\to 0$$

keeping $\rho_{L0}d$ constant.

$$\ln\frac{r^-}{r_2} = \ln\frac{\sqrt{r^2+(d/2)^2 + rd\cos\phi}}{r_2}$$

$$\approx \ln\frac{r}{r_0} + \frac{d}{2r}\cos\phi \quad \text{for } d\to 0 \text{ keeping } \rho_{L0}d \text{ constant}.$$

$$\therefore V = \frac{\rho_{L0}d}{2\pi\epsilon_0 r}\cos\phi.$$

(b) Equipotentials are $\frac{\cos\phi}{r}$ = constant, say, c

or, $\frac{x}{x^2+y^2} = c$ or, $\left(x-\frac{c}{2}\right)^2 + y^2 = \left(\frac{c}{2}\right)^2$

These are cylinders with their axes parallel to the z axis and passing through $x = \pm \frac{c}{2}$ and $y = 0$ and having radii $\frac{c}{2}$.

$$\nabla V = \frac{\rho_{L0} d}{2\pi\epsilon_0} \nabla \left(\frac{\cos\phi}{r}\right) = -\frac{\rho_{L0} d}{2\pi\epsilon_0 r^2} (\cos\phi\, \mathbf{i}_r + \sin\phi\, \mathbf{i}_\phi)$$

Hence, equipotential surfaces are orthogonal to the direction lines.

2.49. By considering an infinitesimal amount of charge in an infinitesimal volume $(dr)(r\,d\theta)(r \sin\theta\, d\phi)$, we obtain $dV = \dfrac{\rho_0 r^2 \sin\theta\, dr\, d\theta\, d\phi}{4\pi\epsilon_0 \sqrt{r^2 + z^2 - 2rz\cos\theta}}$

which upon integration gives

$$V = \int_{r=0}^{a} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} dV = \frac{\rho_0}{2\epsilon_0}\left(a^2 - \frac{r^2}{3}\right) \text{ for } r < a, \text{ and } \frac{\rho_0 a^3}{3\epsilon_0 r} \text{ for } r > a.$$

2.50. Choosing $z = 0$ for the reference plane, we have

$$V = [V]_z - [V]_0 = \int_z^0 \mathbf{E} \cdot d\mathbf{l} = -\int_0^z \mathbf{E} \cdot d\mathbf{l} = -\int_0^z E_z\, dz.$$

It is convenient to perform the integrations graphically. The results are

(a) $-\dfrac{\rho_0 z^2}{2\epsilon_0}$ for $|z| < a$, $\dfrac{\rho_0}{2\epsilon_0}(a^2 - 2a|z|)$ for $|z| > a$

(b) $\dfrac{\rho_0}{2\epsilon_0}\left(2az - \dfrac{z^3}{|z|}\right)$ for $|z| < a$, $\dfrac{\rho_0 a^2 |z|}{2\epsilon_0 z}$ for $|z| > a$

(c) $-\dfrac{|z^3|}{6\epsilon_0}$ for $|z| < a$, $\dfrac{2a^3 - 3a^2|z|}{6\epsilon_0}$ for $|z| > a$

(d) $\dfrac{3a^2 z - z^3}{6\epsilon_0}$ for $|z| < a$, $\dfrac{a^3 |z|}{3\epsilon_0 z}$ for $|z| > a$

(e) $\dfrac{|z^3| - 3az^2}{6\epsilon_0}$ for $|z| < a$, $\dfrac{a^3 - 3a^2|z|}{6\epsilon_0}$ for $|z| > a$

2.51. $V = V(r) - V(0) = \int_r^{r_0} \mathbf{E} \cdot d\mathbf{l} = -\int_{r_0}^{r} E_r\, dr$

For answers, see page 532 of the text. The reference points are (a) $r_0 = a$, (b) $r_0 < a$, and (c) $r_0 = a$.

2.52. $V = V(r) - V(\infty) = \int_r^{\infty} \mathbf{E} \cdot d\mathbf{l} = -\int_{\infty}^{r} E_r\, dr$

(a) $\dfrac{\rho_0}{2\epsilon_0}(b^2 - a^2)$ for $r < a$, $\dfrac{\rho_0}{6\epsilon_0 r}(3b^2 r - r^3 - 2a^3)$ for $a < r < b$, $\dfrac{\rho_0}{3\epsilon_0 r}(b^3 - a^3)$ for $r > b$

(b) $\dfrac{\rho_0}{12\epsilon_0 a}(4a^3 - r^3)$ for $r < a$, $\dfrac{\rho_0 a^3}{4\epsilon_0 r}$ for $r > a$

(c) $\dfrac{\rho_0}{\epsilon_0}\left(\dfrac{a^2}{4} - \dfrac{r^2}{6} + \dfrac{r^4}{20a^2}\right)$ for $r < a$, $\dfrac{2\rho_0 a^3}{15\epsilon_0 r}$ for $r > a$

2.53. (a) From Problem 2.26 (a), $\underset{\sim}{E} = -\frac{\rho_{S0}}{\epsilon_0}\underset{\sim}{i}_z$ for $|z|<a$, and 0 for $|z|>a$.

$$\therefore V = -\int_0^z E_z\,dz = \frac{\rho_{S0} z}{\epsilon_0} \text{ for } |z|<a, \quad \frac{\rho_{S0}\,a|z|}{\epsilon_0 z} \text{ for } |z|>a.$$

(b) From Problem 2.26(c), $\underset{\sim}{E} = 0$ for $r<a$, $\frac{\rho_{S0}a}{\epsilon_0 r}$ for $a<r<b$, and 0 for $r>b$.

$$\therefore V = -\int_b^r E_r\,dr = \frac{\rho_{S0}a}{\epsilon_0}\ln\frac{b}{a} \text{ for } r<a, \quad \frac{\rho_{S0}a}{\epsilon_0}\ln\frac{b}{r} \text{ for } a<r<b, \ 0 \text{ for } r>b.$$

(c) From Problem 2.26(e), $\underset{\sim}{E} = 0$ for $r<a$, $\frac{\rho_{S0}a^2}{\epsilon_0 r^2}$ for $a<r<b$, and 0 for $r>b$.

$$\therefore V = -\int_\infty^r E_r\,dr = \frac{\rho_{S0}a^2}{\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) \text{ for } r<a, \quad \frac{\rho_{S0}a^2}{\epsilon_0}\left(\frac{1}{r}-\frac{1}{b}\right) \text{ for } a<r<b, \ 0 \text{ for } r>b.$$

2.54. $\displaystyle\int_{\text{Vol}} dQ = \rho_0(\text{volume}) = \rho_0\frac{1}{8}\left(\frac{4}{3}\pi a^3\right) = \frac{\rho_0\pi a^3}{6}.$

$$\int_{\text{Vol}} dQ\,\underset{\sim}{r}' = \int_{\text{Vol}} \rho\,\underset{\sim}{r}'\,dv = \rho_0\int_{\text{Vol}} r\,\underset{\sim}{i}_r\,dv$$

$$= \rho_0\int_{r=0}^{a}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2}(r\sin\theta\cos\phi\,\underset{\sim}{i}_x + r\sin\theta\sin\phi\,\underset{\sim}{i}_y + r\cos\theta\,\underset{\sim}{i}_z) \cdot r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \frac{\rho_0\pi a^4}{16}(\underset{\sim}{i}_x + \underset{\sim}{i}_y + \underset{\sim}{i}_z).$$

$$V = \frac{\rho_0\pi a^3}{24\pi\epsilon_0 r} + \frac{\rho_0\pi a^4}{64\pi\epsilon_0 r^2}(\sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta).$$

2.55. (a) $\displaystyle\int dQ\,\underset{\sim}{r}' = \int_{\phi=0}^{2\pi}(\rho_{L0}\cos\phi)(a\,d\phi)(a\cos\phi\,\underset{\sim}{i}_x + a\sin\phi\,\underset{\sim}{i}_y) = \pi a^2\rho_{L0}\,\underset{\sim}{i}_x.$

(b) $\displaystyle\int dQ\,\underset{\sim}{r}' = \int_{\phi=0}^{2\pi}(\rho_{L0}\sin 2\phi)(a\,d\phi)(a\cos\phi\,\underset{\sim}{i}_x + a\sin\phi\,\underset{\sim}{i}_y) = 0$

(c) $\displaystyle\int dQ\,\underset{\sim}{r}' = \int_{\phi=0}^{2\pi}(\rho_{L0}\phi\sin\phi)(a\,d\phi)(a\cos\phi\,\underset{\sim}{i}_x + a\sin\phi\,\underset{\sim}{i}_y) = \frac{\rho_{L0}a^2}{2}(-\pi\,\underset{\sim}{i}_x + 2\pi^2\,\underset{\sim}{i}_y).$

Since $\int dQ$ is zero for cases (a) and (b), dipole moments about any point other than the origin are the same as the dipole moments about the origin for these two cases.

2.56. If the curl of a given vector field is zero, then that field can be realized as a static electric field. Here, all four fields can be realized as static electric fields.

2.57. (a) See Example 2-4 for $\underset{\sim}{E}$ and Example 2-17 for V.

(b) See Problem 2.17. for $\underset{\sim}{E}$ and Problem 2-47 for V.

(c) See Example 2-3 for $\underset{\sim}{E}$ and Example 2-14 for V.

(d) See Problem 2.15. for $\underset{\sim}{E}$ and Problem 2.48 for V.

(e) See Problem 2.11. for $\underset{\sim}{E}$ and Problem 2.46 for V.

(f) See Example 2-6 for $\underset{\sim}{E}$ and Problem 2.49 for V.

2.58. First, we note that $\nabla^2(\frac{1}{r}) = \underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla}(\frac{1}{r}) = 0$ everywhere except at the origin.

Then, we note that

$$\int_{\text{sphere } r\le a} \nabla^2(\tfrac{1}{r})\, dv = \int_{\text{sphere } r\le a} \underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla}(\tfrac{1}{r})\, dv = \int_{\text{surface } r=a} [\underset{\sim}{\nabla}(\tfrac{1}{r})]_{r=a}\cdot \underset{\sim}{i}_r\, ds$$

$$= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \left(-\frac{1}{a^2}\right) a^2 \sin\theta\, d\theta\, d\phi = -4\pi$$

or, $$\lim_{a\to 0} \int_{\text{sphere } r\le a} \nabla^2(\tfrac{1}{r})\, dv = -4\pi.$$

Thus $\nabla^2(\frac{1}{r})$ satisfies the definition of a delta function of strength -4π at the origin. Hence, $\nabla^2(\frac{1}{r}) = -4\pi\,\delta(r)$.

For a point charge Q located at the origin, $\underset{\sim}{\nabla}\cdot\underset{\sim}{E} = \frac{1}{\epsilon_0} Q\,\delta(r)$

or, $$\underset{\sim}{\nabla}\cdot(-\underset{\sim}{\nabla} V) = \frac{1}{\epsilon_0} Q \left[-\frac{1}{4\pi}\nabla^2(\tfrac{1}{r})\right]$$

or, $$\nabla^2 V = \nabla^2\left(\frac{Q}{4\pi\epsilon_0 r}\right)$$

or, $$V = \frac{Q}{4\pi\epsilon_0 r}$$ to within an arbitrary constant which can be assumed to be zero.

CHAPTER 3

3.1. $\mathbf{F}_3 = e\mathbf{v}_3 \times \mathbf{B} = e(\mathbf{v}_1 \times \mathbf{v}_2) \times \dfrac{(\mathbf{F}_2 \times \mathbf{F}_1)}{e(\mathbf{F}_1 \cdot \mathbf{v}_2)} = \dfrac{(\mathbf{v}_1 \times \mathbf{v}_2 \cdot \mathbf{F}_1)\mathbf{F}_2 - (\mathbf{v}_1 \times \mathbf{v}_2 \cdot \mathbf{F}_2)\mathbf{F}_1}{\mathbf{F}_1 \cdot \mathbf{v}_2}$

Subsitituting for $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{v}_1$, and $\mathbf{v}_2$, we get $\mathbf{F}_3 = -e\,\mathbf{i}_x$.

3.2. If r_1 and r_2 are the radii of curvature of the paths followed by m_1 and m_2, respectively, we have $\dfrac{m_1 v^2}{r_1} = q v |\mathbf{B}|$, and $\dfrac{m_2 v^2}{r_2} = q v |\mathbf{B}|$. Thus

$d = 2|r_2 - r_1| = |2(m_2 - m_1)\mathbf{v}| / |q\mathbf{B}|$

3.3. (a) The equations of motion for the test charge are $m\dfrac{dv_x}{dt} = q v_y B_0$, and $m\dfrac{dv_y}{dt} = -q v_x B_0$. Solving these equations and using the initial conditions $v_x = 0$ and $v_y = v_0$ for $t = 0$, and $x = 0$ and $y = 0$ for $t = 0$, we get $x = \dfrac{v_0}{\omega_c}(1 - \cos\omega_c t)$ and $y = \dfrac{v_0}{\omega_c}\sin\omega_c t$ where $\omega_c = \dfrac{qB_0}{m}$. We can express this solution as $\left(x - \dfrac{v_0}{\omega_c}\right)^2 + y^2 = \left(\dfrac{v_0}{\omega_c}\right)^2$. This is the equation of a circle.

(b) The test charge emerges from the field region at a time t_1 given by

$L = \dfrac{v_0}{\omega_c}\sin\omega_c t_1$ or $\sin\omega_c t_1 = \dfrac{\omega_c L}{v_0}$. Hence,

$$x_L = [x]_{t=t_1} = \frac{v_0}{\omega_c}\left[1 - \sqrt{1 - \left(\frac{\omega_c L}{v_0}\right)^2}\,\right]$$

$$\mathbf{v}_L = \left[\frac{dx}{dt}\mathbf{i}_x + \frac{dy}{dt}\mathbf{i}_y\right]_{t=t_1} = \omega_c L\,\mathbf{i}_x + v_0\sqrt{1 - \left(\frac{\omega_c L}{v_0}\right)^2}\;\mathbf{i}_y.$$

(c) Once the charge emerges from the field region, it follows a straight line path along the direction of $\mathbf{v}_L$. Since the time taken by the charge to reach the $y = L + d$ plane from the $y = L$ plane is $d\Big/v_0\sqrt{1 - \left(\frac{\omega_c L}{v_0}\right)^2}$, we obtain $x_d = x_L + \left[\omega_c L d\Big/v_0\sqrt{1 - \left(\frac{\omega_c L}{v_0}\right)^2}\,\right]$.

3.4. The equations of motion of the electron are

$$m\frac{dv_x}{dt} = e v_y B_0, \quad m\frac{dv_y}{dt} = -e v_x B_0, \text{ and } m\frac{dv_z}{dt} = 0.$$

Solving these equations and using the initial conditions $v_x = v_{x0}$, $v_y = v_{y0}$, and $v_z = v_{z0}$ for $t = 0$, and $x = y = z = 0$ for $t = 0$, we get

$$x = \frac{1}{\omega_c}\left(v_{x0}\sin\omega_c t - v_{y0}\cos\omega_c t + v_{y0}\right)$$

$$y = \frac{1}{\omega_c}\left(v_{y0}\sin\omega_c t + v_{x0}\cos\omega_c t - v_{x0}\right)$$

$$z = v_{z0}\,t$$

which can be written as

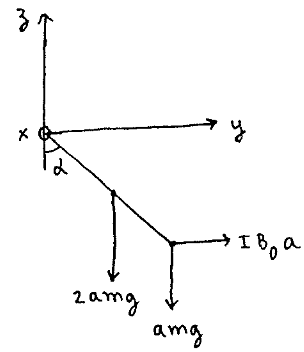
$$\left(x - \frac{v_{y0}}{\omega_c}\right)^2 + \left(y + \frac{v_{x0}}{\omega_c}\right)^2 = \frac{v_{x0}^2 + v_{y0}^2}{\omega_c^2} \quad \text{and} \quad z = v_{z0}\,t$$

where $\omega_c = \dfrac{eB_0}{m}$. These are the equations for a helix of radius $\dfrac{1}{|\omega_c|}\sqrt{v_{x0}^2 + v_{y0}^2}$. One turn of the helix is completed in a period $T = \dfrac{2\pi}{\omega_c}$ and hence the pitch of the helix is $\dfrac{2\pi|v_{z0}|}{|\omega_c|}$.

3.5. Equating the magnetic force with the gravitational force, we have $ILB_0 = mg$ or, $I = \dfrac{mg}{LB_0}$. The current must flow from west to east. For $L = 1\,m$, $m = 30$ gms, $B_0 = 0.3\times10^{-4}\ wb/m^2$, and $g = 9.8\ m/sec^2$,

$I = 9800$ amp.

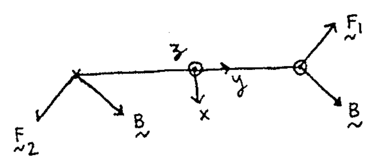
3.6. If α is the angle by which the loop swings from the vertical, we obtain by equating the torques



$$IB_0 a\cos\alpha\,(a) = amg\sin\alpha\,(a) + 2amg\sin\alpha\left(\frac{a}{2}\right)$$

$$\tan\alpha = \frac{IB_0}{2mg} \quad \text{or,} \quad \alpha = \tan^{-1}\frac{IB_0}{2mg}.$$

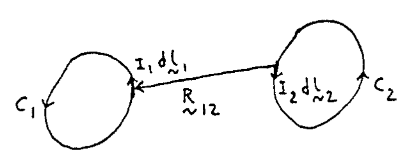
3.7. Denoting the dimensions of the loop to be l parallel to the z axis and w perpendicular to it, we have



$$\mathbf{F}_1 = Il\,\mathbf{i}_z\times\mathbf{B}, \quad \mathbf{F}_2 = -Il\,\mathbf{i}_z\times\mathbf{B}.$$

$$\text{Torque} = \mathbf{F}_1\cdot(-\mathbf{i}_x)\,w\,\mathbf{i}_z = -Ilw(\mathbf{i}_z\times\mathbf{B}\cdot\mathbf{i}_x)\,\mathbf{i}_z = IA(\mathbf{i}_z\times\mathbf{i}_x\cdot\mathbf{B})\,\mathbf{i}_z$$
$$= IA(\mathbf{i}_y\cdot\mathbf{B})\,\mathbf{i}_z.$$

3.8. Force experienced by loop C_1 is given by



$$\mathbf{F}_1 = \frac{\mu_0}{4\pi}\oint_{C_1}\oint_{C_2}\frac{I_1 d\mathbf{l}_1\times(I_2 d\mathbf{l}_2\times\mathbf{R}_{12})}{R_{12}^3}$$

$$= \frac{\mu_0 I_1 I_2}{4\pi}\left[\oint_{C_1}\oint_{C_2}\frac{(d\mathbf{l}_1\cdot\mathbf{R}_{12})\,d\mathbf{l}_2}{R_{12}^3} - \oint_{C_1}\oint_{C_2}\frac{(d\mathbf{l}_1\cdot d\mathbf{l}_2)\,\mathbf{R}_{12}}{R_{12}^3}\right]$$

But $\oint_{C_1}\oint_{C_2}\dfrac{(d\boldsymbol{l}_1\cdot\mathbf{R}_{12})\,d\boldsymbol{l}_2}{R_{12}^3} = -\oint_{C_2}\oint_{C_1}\left(d\boldsymbol{l}_1\cdot\nabla_1\dfrac{1}{R_{12}}\right)d\boldsymbol{l}_2 = 0$

since $\oint_{C_1} d\boldsymbol{l}_1\cdot\nabla_1\dfrac{1}{R_{12}} = \int_{S_1}\left(\nabla_1\times\nabla_1\dfrac{1}{R_{12}}\right)\cdot d\mathbf{S} = 0$ where S_1 is any surface bounded by C_1. Thus $\mathbf{F}_1 = -\dfrac{\mu_0 I_1 I_2}{4\pi}\oint_{C_1}\oint_{C_2}\dfrac{(d\boldsymbol{l}_1\cdot d\boldsymbol{l}_2)\,\mathbf{R}_{12}}{R_{12}^3}$.

Similarly, the force $\mathbf{F}_2$ experienced by loop C_2 given by

$\mathbf{F}_2 = \dfrac{\mu_0}{4\pi}\oint_{C_2}\oint_{C_1}\dfrac{I_2\,d\boldsymbol{l}_2\times(I_1\,d\boldsymbol{l}_1\times\mathbf{R}_{21})}{R_{21}^3}$ can be shown to be equal to

$-\dfrac{\mu_0 I_1 I_2}{4\pi}\oint_{C_1}\oint_{C_2}\dfrac{(d\boldsymbol{l}_2\cdot d\boldsymbol{l}_1)\,\mathbf{R}_{21}}{R_{21}^3} = -\mathbf{F}_1$ since $\mathbf{R}_{21} = -\mathbf{R}_{12}$.

3.9. $\mathbf{F}_{21} = \dfrac{\mu_0}{4\pi} I_2\,dl_2\,\mathbf{i}_y\times[I_1\,dl_1\,\mathbf{i}_y\times\mathbf{i}_z]/(1)^3 = -\dfrac{\mu_0}{4\pi}I_1 I_2\,dl_1\,dl_2\,\mathbf{i}_z$.

$\mathbf{F}_{12} = \dfrac{\mu_0}{4\pi} I_1\,dl_1\,\mathbf{i}_y\times[I_2\,dl_2\,\mathbf{i}_y\times(-\mathbf{i}_z)]/(1)^3 = \dfrac{\mu_0}{4\pi}I_1 I_2\,dl_1\,dl_2\,\mathbf{i}_z$.

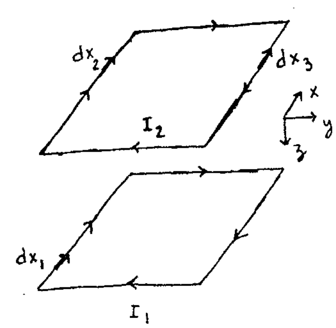
$\mathbf{F}_{31} = \dfrac{\mu_0}{4\pi} I_2\,dl_3(-\mathbf{i}_x)\times[I_1\,dl_1\,\mathbf{i}_y\times(-\mathbf{i}_x+\mathbf{i}_y+\mathbf{i}_z)]/(\sqrt{3})^3 = \dfrac{\mu_0}{4\pi}I_1 I_2\,dl_1\,dl_3\,\mathbf{i}_y/3\sqrt{3}$.

$\mathbf{F}_{13} = \dfrac{\mu_0}{4\pi} I_1\,dl_1\,\mathbf{i}_y\times[I_2\,dl_3(-\mathbf{i}_x)\times(\mathbf{i}_x-\mathbf{i}_y-\mathbf{i}_z)]/(\sqrt{3})^3 = \dfrac{\mu_0}{4\pi}I_1 I_2\,dl_1\,dl_3\,\mathbf{i}_x/3\sqrt{3}$.

$\mathbf{F}_{32} = \dfrac{\mu_0}{4\pi} I_2\,dl_3(-\mathbf{i}_x)\times[I_2\,dl_2\,\mathbf{i}_y\times(-\mathbf{i}_x+\mathbf{i}_y)]/(\sqrt{2})^3 = \dfrac{\mu_0}{4\pi}I_2^2\,dl_2\,dl_3\,\mathbf{i}_y/2\sqrt{2}$.

$\mathbf{F}_{23} = \dfrac{\mu_0}{4\pi} I_2\,dl_2\,\mathbf{i}_y\times[I_2\,dl_3(-\mathbf{i}_x)\times(\mathbf{i}_x-\mathbf{i}_y)]/(\sqrt{2})^3 = \dfrac{\mu_0}{4\pi}I_2^2\,dl_2\,dl_3\,\mathbf{i}_x/2\sqrt{2}$.

3.10. From Problem 3.8., the force experienced by closed loop 1 due to closed loop 2 is



$$\mathbf{F}_1 = -\frac{\mu_0 I_1 I_2}{4\pi}\oint_{C_1}\oint_{C_2}\frac{(d\boldsymbol{l}_1\cdot d\boldsymbol{l}_2)\,\mathbf{R}_{12}}{R_{12}^3}$$

Hence, pairs of sides which are perpendicular to each other do not contribute to $\mathbf{F}_1$. With reference to the notation shown in the figure,

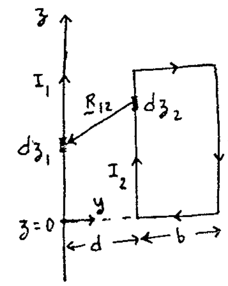
$$\frac{\mathbf{F}_1}{4} = -\frac{\mu_0 I_1 I_2}{4\pi}\int_{x_2=0}^{a}\int_{x_1=0}^{a}\frac{dx_1\,dx_2\,[(x_1-x_2)\,\mathbf{i}_x + d\,\mathbf{i}_z]}{[(x_1-x_2)^2+d^2]^{3/2}}$$

$$+\frac{\mu_0 I_1 I_2}{4\pi}\int_{x_3=0}^{a}\int_{x_1=0}^{a}\frac{dx_1\,dx_3\,[(x_1-x_3)\,\mathbf{i}_x - a\,\mathbf{i}_y + d\,\mathbf{i}_z]}{[(x_1-x_3)^2+a^2+d^2]^{3/2}}$$

Only the z component of $\mathbf{F}_1/4$ is of interest since the x and y components of the forces on opposite sides are equal and opposite and hence cancel. Evaluating the z component, we get

$$\mathbf{F}_1 = -\frac{2\mu_0 I_1 I_2}{\pi}\left[\frac{1}{d}\left(\sqrt{a^2+d^2}-d\right) - \frac{d}{a^2+d^2}\left(\sqrt{2a^2+d^2}-\sqrt{a^2+d^2}\right)\right]\mathbf{i}_z .$$

3.11. $$\mathbf{F}_1 = \frac{\mu_0}{4\pi}\int_{z=-\infty}^{\infty} I_1\, dz\, \mathbf{i}_z \times \oint_{C_2} \frac{I_2\, d\mathbf{l}_2 \times \mathbf{R}_{12}}{R_{12}^3}$$



$$\oint_{C_2}\frac{d\mathbf{l}_2\times\mathbf{R}_{12}}{R_{12}^3} = \int_{z_2=0}^{a}\frac{dz_2\,\mathbf{i}_z\times[-d\,\mathbf{i}_y+(z_1-z_2)\,\mathbf{i}_z]}{[d^2+(z_1-z_2)^2]^{3/2}}$$

$$+\int_{y=d}^{d+b}\frac{dy\,\mathbf{i}_y\times[-y\,\mathbf{i}_y+(z_1-a)\,\mathbf{i}_z]}{[y^2+(z_1-a)^2]^{3/2}}$$

$$-\int_{z_2=0}^{a}\frac{dz_2\,\mathbf{i}_z\times[-(d+b)\,\mathbf{i}_y+(z_1-z_2)\,\mathbf{i}_z]}{[(d+b)^2+(z_1-z_2)^2]^{3/2}}$$

$$-\int_{y=d}^{d+b}\frac{dy\,\mathbf{i}_y\times[-y\,\mathbf{i}_y+z_1\,\mathbf{i}_z]}{[y^2+z_1^2]^{3/2}}$$

Evaluating $\oint_{C_2}\frac{d\mathbf{l}_2\times\mathbf{R}_{12}}{R_{12}^3}$ and then evaluating $\mathbf{F}_1$, we get

$$\mathbf{F}_1 = \frac{\mu_0 I_1 I_2 a}{2\pi}\left(\frac{1}{d}-\frac{1}{d+b}\right)\mathbf{i}_y .$$

$$\mathbf{F}_2 = \frac{\mu_0}{4\pi}\oint_{C_2} I_2\, d\mathbf{l}_2 \times \int_{z_1=-\infty}^{\infty}\frac{I_1\, dz_1\,\mathbf{i}_z\times\mathbf{R}_{21}}{R_{21}^3}$$

$$\int_{z_1=-\infty}^{\infty}\frac{dz_1\,\mathbf{i}_z\times\mathbf{R}_{21}}{R_{21}^3} = \int_{z_1=-\infty}^{\infty}\frac{dz_1\,\mathbf{i}_z\times[y\,\mathbf{i}_y+(z-z_1)\,\mathbf{i}_z]}{[y^2+(z-z_1)^2]^{3/2}} = -\frac{2}{y}\,\mathbf{i}_x .$$

$$\mathbf{F}_2 = \frac{\mu_0 I_2 I_1}{4\pi}\oint_{C_2} d\mathbf{l}_2\times\left[-\frac{2}{y}\,\mathbf{i}_x\right]$$

$$= -\frac{\mu_0 I_2 I_1}{4\pi}\left[\int_{z_2=0}^{a} dz_2\,\mathbf{i}_z\times\frac{2}{d}\,\mathbf{i}_x + \int_{y=d}^{d+b} dy\,\mathbf{i}_y\times\frac{2}{y}\,\mathbf{i}_x\right.$$

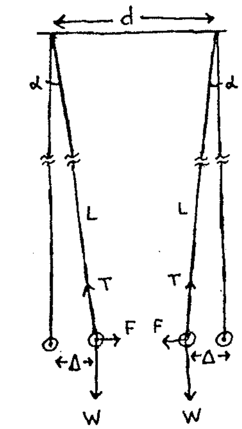
$$\left. -\int_{z_2=0}^{a} dz_2\,\mathbf{i}_z\times\frac{2}{d+b}\,\mathbf{i}_x - \int_{y=d}^{d+b} dy\,\mathbf{i}_y\times\frac{2}{y}\,\mathbf{i}_x\right]$$

$$= -\frac{\mu_0 I_2 I_1 a}{4\pi}\left(\frac{1}{d}-\frac{1}{d+b}\right)\mathbf{i}_y = -\mathbf{F}_1 .$$

3.12. (a) The magnetic field acting on the wire occupying the line $x=1, y=1$ is $\frac{\mu_0}{4\pi}(-\mathbf{i}_x+\mathbf{i}_y) - \frac{\mu_0}{2\pi}\mathbf{i}_x + \frac{\mu_0}{2\pi}\mathbf{i}_y = \frac{3\mu_0}{4\pi}(-\mathbf{i}_x+\mathbf{i}_y)$. Force per unit length of that wire $= -\frac{3\mu_0}{4\pi}(\mathbf{i}_x+\mathbf{i}_y)$. Thus the force on each wire is $0.3377\,\mu_0$ per unit length and is directed towards the opposite wire.

(b) $0.2149\,\mu_0(-\underset{\sim}{i}_x + \underset{\sim}{i}_y)$

(c) $\mu_0(-0.382\,\underset{\sim}{i}_x - 0.114\,\underset{\sim}{i}_y)$



3.13. For the wires to attract, the currents must flow in the same direction. Since $d \ll l$, the wires can be considered to be infinitely long. Let the swing from the vertical position be Δ as shown in the figure. Then, the force on each rod is $\dfrac{I^2 l}{2\pi(d-2\Delta)}$. For equilibrium,

$$\tan\alpha = \frac{\Delta}{L} = \frac{F}{W} = \frac{I^2 l}{2\pi(d-2\Delta)W}$$

or, $\dfrac{I^2 lL}{2W} = (d-2\Delta)\,\Delta$. For $I = 0$, $\Delta = 0$. As I increases, Δ increases, $(d-2\Delta)$ decreases and hence there is a maximum value for $(d-2\Delta)\Delta$ beyond which the equation cannot be satisfied. This maximum value occurs for $\Delta = \frac{d}{4}$, that is, for $I = \frac{d}{2}\sqrt{\frac{\pi W}{lL}}$. For any further increase in current, the force F becomes greater than $T\sin\alpha$ and hence the rods swing and touch each other.

3.14. The magnetic field at $(0,0,z)$ has a z component only since the radial components due to oppositely situated pairs of current elements on the loop cancel. Hence

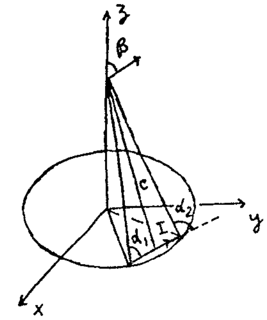
$$\underset{\sim}{B} = \left[\int_{\phi=0}^{2\pi} dB_z\right]\underset{\sim}{i}_z = \left[\int_{\phi=0}^{2\pi} \frac{\mu_0 I a\,d\phi}{4\pi(a^2+z^2)}\cdot\frac{a}{\sqrt{a^2+z^2}}\right]\underset{\sim}{i}_z = \frac{\mu_0 I a^2}{2(a^2+z^2)^{3/2}}\underset{\sim}{i}_z.$$

For $z \to 0$, $\underset{\sim}{B} = \displaystyle\int_{\phi=0}^{2\pi} \frac{\mu_0 I\,d\phi}{4\pi a}\,\underset{\sim}{i}_z = \frac{\mu_0 I}{2a}\,\underset{\sim}{i}_z$ which is the same as the above result tends to.

3.15. From symmetry considerations, $\underset{\sim}{B}$ at a point on the z axis has a z component only. Hence, it is sufficient if we compute B_z due to one side of the polygon and multiply it by n. With reference to the notation shown in the figure, this is given by

$$\Delta B_z = \frac{\mu_0 I}{4\pi c}(\cos\alpha_1 - \cos\alpha_2)\cos\beta$$

$$= \frac{\mu_0 I}{2\pi c^2} \frac{a^2 \sin\frac{\pi}{n} \cos\frac{\pi}{n}}{\sqrt{\left(a \sin\frac{\pi}{n}\right)^2 + c^2}}$$

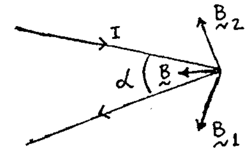


$$\mathbf{B} = (n\, \Delta B_z)\, \mathbf{i}_z = \frac{n \mu_0 I a^2 \sin\frac{2\pi}{n}}{4\pi\left[\left(a \cos\frac{\pi}{n}\right)^2 + z^2\right]\left[a^2+z^2\right]^{1/2}} \mathbf{i}_z$$

For $n \to \infty$, $n \sin\frac{2\pi}{n} \to 2\pi$, $\cos\frac{\pi}{n} \to 1$,

$\mathbf{B} \to \dfrac{\mu_0 I a^2}{2(a^2+z^2)^{3/2}} \mathbf{i}_z$ which agrees with the result of Problem 3.14.

3.16. $B_1 = B_2 = \dfrac{\mu_0 I}{4\pi d}(\cos 0° - \cos 90°) = \dfrac{\mu_0 I}{4\pi d}$.



$B = B_1 \sin\frac{\alpha}{2} + B_2 \sin\frac{\alpha}{2} = \dfrac{\mu_0 I}{2\pi d} \sin\frac{\alpha}{2}$.

$\mathbf{B} = \dfrac{\mu_0 I}{2\pi d} \sin\frac{\alpha}{2}\, \mathbf{i}_\alpha$ where $\mathbf{i}_\alpha$ is the unit vector along the bisector of angle α. For $\alpha = \pi$, $\sin\frac{\alpha}{2} = 1$, $\mathbf{B} = \dfrac{\mu_0 I}{2\pi d} \mathbf{i}_\phi$ where $\mathbf{i}_\phi$ is circular to the wire.

3.17. (a) Using the result of Problem 3.14, we obtain

$$\mathbf{B} = \frac{\mu_0 I a^2}{2} \left\{ \frac{1}{[a^2 + (z-b)^2]^{3/2}} + \frac{1}{[a^2+(z+b)^2]^{3/2}} \right\} \mathbf{i}_z$$

(b) $\left[\dfrac{dB_z}{dz}\right]_{z=0} \equiv 0$; $\left[\dfrac{d^2 B_z}{dz^2}\right]_{z=0} = -3\mu_0 I a^2 \left[(a^2+b^2)^{5/2} - 5b^2(a^2+b^2)^{-7/2}\right]$

$\left[\dfrac{d^2 B_z}{dz^2}\right]_{z=0} = 0$ for $b = \dfrac{a}{2}$; $\left[\dfrac{d^3 B_z}{dz^3}\right]_{z=0} \equiv 0$.

3.18. The magnetic flux density at $(0,0,z)$ due to a ring of current formed by width dz' of the solenoid at $z = z'$ is given, from the result of Problem 3.14, by $d\mathbf{B} = \dfrac{\mu_0 (nI\, dz')\, a^2}{2[a^2 + (z-z')^2]^{3/2}} \mathbf{i}_z$. The magnetic flux density due to the entire solenoid is then given by

$$\mathbf{B} = \int_{z'=-L_1}^{L_2} d\mathbf{B} = \frac{\mu_0 n I}{2} \left[\frac{z+L_1}{\sqrt{a^2+(z+L_1)^2}} - \frac{z-L_2}{\sqrt{a^2+(z-L_2)^2}} \right]$$

For $L_1, L_2 \to \infty$, $\mathbf{B} \to \mu_0 n I\, \mathbf{i}_z$ which is the same as that for an infinitely long solenoid.

3.19. We consider a ring of radius r and width dr and obtain the magnetic field due to it at $(0,0,z)$ as

$$d\underset{\sim}{B} = \frac{\mu_0 (nI\,dr)\, r^2}{2(r^2+z^2)^{3/2}}\, \underset{\sim}{i}_z$$ from the result of Problem 3.14, which gives the required integral for $\underset{\sim}{B}$. For answers to (a), (b), and (c), see page 533 of the text.

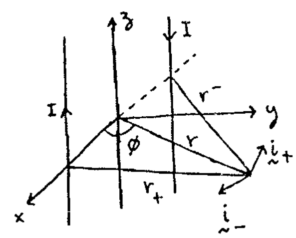
3.20. We consider a ring of width $a\,d\theta'$ at $\theta = \theta'$ and obtain the magnetic field due to it at $(0,0,z)$ as

$$d\underset{\sim}{B} = \frac{\mu_0 (nIa\,d\theta')(a\sin\theta')^2}{2[(a\sin\theta')^2 + (z - a\cos\theta')^2]^{3/2}}\, \underset{\sim}{i}_z$$ from the result of Problem 3.14, which then gives the required integral for $\underset{\sim}{B}$.

(a) $\dfrac{2\mu_0 n_0 I}{3}\, \underset{\sim}{i}_z$ for $|z| < a$, $\dfrac{2\mu_0 n_0 I a^3}{3|z^3|}\, \underset{\sim}{i}_z$ for $|z| > a$

(b) $\dfrac{\mu_0 n_0 I a^2}{(a^2 - z^2)}\, \underset{\sim}{i}_z$ for $|z| < a$, $\dfrac{\mu_0 n_0 I a^3}{|z|(z^2 - a^2)}\, \underset{\sim}{i}_z$ for $|z| > a$

3.21. (a)



$$\underset{\sim}{B} = \frac{\mu_0 I}{2\pi r_+}\, \underset{\sim}{i}_+ + \frac{\mu_0 I}{2\pi r_-}\, \underset{\sim}{i}_-$$

$$= \frac{\mu_0 I}{2\pi r_+^2}\left[-\frac{d}{2}\sin\phi\, \underset{\sim}{i}_r + \left(r - \frac{d}{2}\cos\phi\right)\underset{\sim}{i}_\phi\right]$$

$$+ \frac{\mu_0 I}{2\pi r_-^2}\left[-\frac{d}{2}\sin\phi\, \underset{\sim}{i}_r - \left(r + \frac{d}{2}\cos\phi\right)\underset{\sim}{i}_\phi\right]$$

where $r_+ = \left[r^2 + \left(\frac{d}{2}\right)^2 - rd\cos\phi\right]^{1/2}$ and $r_- = \left[r^2 + \left(\frac{d}{2}\right)^2 + rd\cos\phi\right]^{1/2}$

$$\underset{\sim}{B} = -\frac{\mu_0 I d\sin\phi}{2\pi}\left[\frac{r^2 + (d/2)^2}{r_+^2 r_-^2}\right]\underset{\sim}{i}_r + \frac{\mu_0 I d\cos\phi}{2\pi}\left[\frac{r^2 - (d/2)^2}{r_+^2 r_-^2}\right]\underset{\sim}{i}_\phi$$

$$\rightarrow \frac{\mu_0 I d}{2\pi r^2}(-\sin\phi\, \underset{\sim}{i}_r + \cos\phi\, \underset{\sim}{i}_\phi)$$ for $d \to 0$ keeping Id constant.

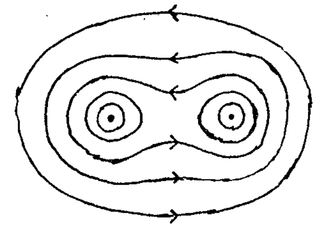
(b) Substituting for B_r, B_ϕ, and B_z in $\dfrac{dr}{B_r} = \dfrac{r\,d\phi}{B_\phi} = \dfrac{dz}{0}$ and simplifying, we obtain $d\left(-\dfrac{\cos\phi}{r}\right) = 0$ and $dz = 0$, or, $-\dfrac{\cos\phi}{r} = $ constant and $z = $ constant, which can be written as $-\dfrac{x}{x^2+y^2} = c$ and $z = $ constant, or $\left(x - \frac{c}{2}\right)^2 + y^2 = \left(\frac{c}{2}\right)^2$ and $z = $ constant. These are circles in the planes $z = $ constant, with centers at $x = \frac{c}{2}$, $y = 0$ and radii equal to $\dfrac{|c|}{2}$.

3.22. Derivation similar to that in Problem 3.21(a) except that it is convenient to work in cartesian coordinates. Thus

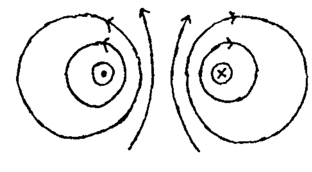
$$\mathbf{B} = \frac{\mu_0 I_1}{2\pi\left[(x-\frac{d}{2})^2+y^2\right]}\left[(x-\tfrac{d}{2})\mathbf{i}_y - y\,\mathbf{i}_x\right] + \frac{\mu_0 I_2}{2\pi\left[(x+\frac{d}{2})^2+y^2\right]}\left[(x+\tfrac{d}{2})\mathbf{i}_y - y\,\mathbf{i}_x\right]$$

Finding B_x and B_y and substituting in $\frac{dx}{B_x} = \frac{dy}{B_y}$ and cross multiplying and integrating, we get the given equation for the direction lines of $\mathbf{B}$.

(a) $\left[(x+\frac{d}{2})^2+y^2\right]\left[(x-\frac{d}{2})^2+y^2\right] = \text{constant}$



(b) $\dfrac{(x+\frac{d}{2})^2+y^2}{(x-\frac{d}{2})^2+y^2} = \text{constant, say, } c^2.$



Cross multiplying and rearranging, we get

$$\left(x - \frac{d}{2}\,\frac{c^2+1}{c^2-1}\right)^2 + y^2 = \left(\frac{dc}{c^2-1}\right)^2$$

which represent circles.

3.23. (a) We use the result of Problem 3.14. to obtain

$$\mathbf{B}(0,0,z) = \frac{\mu_0 I a^2}{2}\left\{\frac{1}{[a^2+(z-d)^2]^{3/2}} - \frac{1}{[a^2+(z+d)^2]^{3/2}}\right\}\mathbf{i}_z$$

which reduces to $\dfrac{3\mu_0 I a^2 d}{z^4}\mathbf{i}_z$ for $z \gg a$ and d.

(b) Following the method of derivation employed in Example 3-5, we obtain the total field at a point on the y axis due to four current elements symmetrically situated about it making angles ϕ' with the x axis as

$$d\mathbf{B} = -\frac{\mu_0 I a d\,(y^2+d^2)\sin\phi'}{\pi(y^2+d^2)\left[a^2+y^2+d^2-2ay\sin\phi'\right]^{3/2}}\,d\phi'\,\mathbf{i}_y$$

$$\approx \frac{\mu_0 I a d}{\pi y^3}\left(\sin\phi' + \frac{3a}{y}\sin^2\phi'\right)\mathbf{i}_y \quad \text{for } y \gg a \text{ and } d.$$

Integrating $d\mathbf{B}$ from $\phi' = -\frac{\pi}{2}$ to $\phi' = \frac{\pi}{2}$ and replacing y by r, we obtain $\mathbf{B} = -\dfrac{3\mu_0 I a^2 d}{2r^4}\mathbf{i}_r$.

3.24. We consider a width dx of the surface current at a distance x from the z axis and write the magnetic field due to it at $(0,y,0)$ as

$$d\mathbf{B} = \frac{\mu_0 J_S\,dx}{2\pi\sqrt{x^2+y^2}}\left[-\frac{y}{\sqrt{x^2+y^2}}\mathbf{i}_x - \frac{x}{\sqrt{x^2+y^2}}\mathbf{i}_y\right]$$

which gives the required integrals for B_x and B_y.

(a) $B_x = -\dfrac{\mu_0 J_{s0}}{\pi} \tan^{-1}\dfrac{a}{y}$, $B_y = 0$, $B_z = 0$

(b) $B_x = -\dfrac{\mu_0 J_{s0}}{\pi}\left(\tan^{-1}\dfrac{a}{y} - \dfrac{y}{2a}\ln\dfrac{a^2+y^2}{y^2}\right)$, $B_y = 0$, $B_z = 0$

(c) $B_x = 0$, $B_y = -\dfrac{\mu_0 J_{s0}}{\pi}\left(1 - \dfrac{y}{a}\tan^{-1}\dfrac{a}{y}\right)$, $B_z = 0$

3.25. We consider two current elements $J_s\,(dr')(r'd\phi')\,\underset{\sim}{i}_r$ at the points (r', ϕ') and $(r', -\phi')$ symmetrically situated about the xz plane and obtain the magnetic field due to them at a point $P(x, 0, z)$ as

$$d\underset{\sim}{B} = -\frac{\mu_0 I}{8\pi^2}\,\frac{2z\cos\phi'\,dr'\,d\phi'}{[x^2 + r'^2 + z^2 - 2r'x\cos\phi']^{3/2}}\,\underset{\sim}{i}_y$$ which then gives

$$\underset{\sim}{B} = \int_{\phi'=0}^{\pi}\int_{r'=0}^{\infty} d\underset{\sim}{B} = \frac{\mu_0 I}{4\pi}\left[\frac{z}{x\sqrt{x^2+z^2}} - \frac{z}{|z|\,x}\right]\underset{\sim}{i}_y$$

both above and below the xy plane.

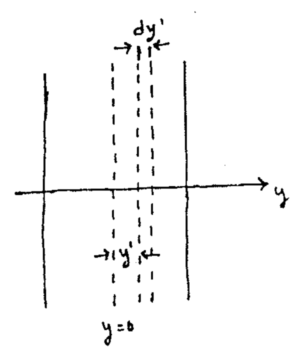
From Example 3-3, the magnetic field at point P above the xy plane due to a filamentary wire along the negative z axis carrying current I from the origin to $z = -\infty$ is

$$\underset{\sim}{B} = -\frac{\mu_0 I}{4\pi x}\left[-\frac{z}{\sqrt{x^2+z^2}} + 1\right]\underset{\sim}{i}_y$$ which agrees with the above expression.

For a filamentary wire along the positive z axis carrying current I from the origin to $z = \infty$, the magnetic field at point P below the xy plane is

$$\underset{\sim}{B} = \frac{\mu_0 I}{4\pi x}\left[\frac{z}{\sqrt{x^2+z^2}} + 1\right]\underset{\sim}{i}_y$$ which agrees with the expression derived for the radial current distribution.

3.26. We divide the volume current into a series of slabs of infinitesimal thickness dy'. Let us consider one such slab located at $y = y'$. We then have



$$d\underset{\sim}{B} = \begin{cases} -\dfrac{\mu_0 J\,dy'}{2}\,\underset{\sim}{i}_x & \text{for } y > y' \\[2ex] \dfrac{\mu_0 J\,dy'}{2}\,\underset{\sim}{i}_x & \text{for } y < y' \end{cases}$$

which gives

$$
B_x = \begin{cases} -\displaystyle\int_{y'=-a}^{a} \frac{\mu_0 J\,dy'}{2} = -\frac{\mu_0}{2}\int_{y=-a}^{a} J\,dy & \text{for } y>a \\ -\displaystyle\int_{y'=-a}^{y} \frac{\mu_0 J\,dy'}{2} + \int_{y'=y}^{a} \frac{\mu_0 J\,dy'}{2} = \frac{\mu_0}{2}\left[\int_{y=y}^{a} J\,dy - \int_{y=-a}^{y} J\,dy\right] & \text{for } -a<y<a \\ \displaystyle\int_{y'=-a}^{a} \frac{\mu_0 J\,dy'}{2} = \frac{\mu_0}{2}\int_{y=-a}^{a} J\,dy & \text{for } y<-a \end{cases}
$$

(a) $-\mu_0 J_0 y$ for $|y|<a$, $-\mu_0 J_0 a \frac{|y|}{y}$ for $|y|>a$

(b) $\mu_0 J_0 (|y|-a)$ for $|y|<a$, 0 for $|y|>a$

(c) $-\frac{\mu_0 y^3}{2|y|}$ for $|y|<a$, $-\frac{\mu_0 a^2 |y|}{2y}$ for $|y|>a$

(d) $\frac{\mu_0}{2}(a^2-y^2)$ for $|y|<a$, 0 for $|y|>a$

3.27. We consider the filamentary currents corresponding to infinitesimal areas $r'\,dr'\,d\phi'$ at (r', ϕ') and $(r', -\phi')$ and obtain the magnetic field due to them at $(x, 0, z)$ as

$$d\mathbf{B} = \frac{\mu_0 J r'\,dr'\,d\phi'\,(x - r'\cos\phi')}{\pi\,[r'^2 + x^2 - 2r'x\cos\phi']}\,\mathbf{i}_y \quad \text{which then gives}$$

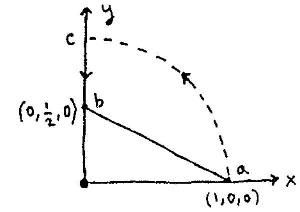
$$\mathbf{B} = \int_{r'=0}^{a}\int_{\phi'=0}^{\pi} d\mathbf{B} = \frac{\mu_0}{x}\int_{r'=0}^{r} J r'\,dr'\,\mathbf{i}_y$$

By setting $x = r$ in view of the cylindrical symmetry and replacing the variable of integration r' by r, we obtain the given integral. For answers to (a), (b), and (c), see page 533 of the text.

3.28. $\mathbf{B}\cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi r}\,\mathbf{i}_\phi \cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi(x^2+y^2)}(-y\,dx + x\,dy)$

(a) Along path $x + 2y = 1$ and $z = 0$, $x = 1 - 2y$, $dx = -2\,dy$

$$\mathbf{B}\cdot d\mathbf{l} = \frac{\mu_0 I\,dy}{2\pi(5y^2 - 4y + 1)}$$



$$\int_{1,0,0}^{0,\frac{1}{2},0} \mathbf{B}\cdot d\mathbf{l} = \frac{\mu_0 I}{4}.$$

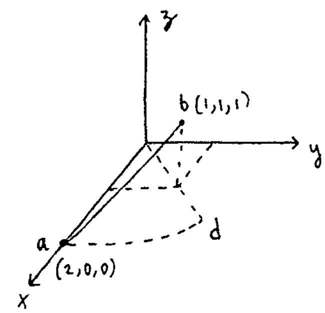
From considerations of symmetry and Ampere's circuital law,

$$\int_a^b \mathbf{B}\cdot d\mathbf{l} = \int_a^c \mathbf{B}\cdot d\mathbf{l} + \int_c^b \mathbf{B}\cdot d\mathbf{l} = \frac{\mu_0 I}{4} + 0 = \frac{\mu_0 I}{4}.$$

(b) Equation for the path is $y = 2-x$, $z = 2-x$. $\therefore dy = -dx$.

$$\underset{\sim}{B}\cdot d\underset{\sim}{l} = -\frac{\mu_0 I}{2\pi}\frac{dx}{x^2-2x+2}$$

$$\int_{2,0,0}^{1,1,1} \underset{\sim}{B}\cdot d\underset{\sim}{l} = \frac{\mu_0 I}{8}.$$



From considerations of symmetry and Ampere's circuital law,

$$\int_a^b \underset{\sim}{B}\cdot d\underset{\sim}{l} = \int_a^d \underset{\sim}{B}\cdot d\underset{\sim}{l} = \frac{\mu_0 I}{8}.$$

3.29. The solution for each part consists of choosing an appropriate closed path from considerations of symmetry and then applying Ampere's circuital law. For answers, see page 533 of the text. The closed paths are as follows:

(a) Rectangular path in z = constant plane with one side in the plane $y=0$.

(b) Rectangular path in z = constant plane with one side in the plane $y=a$ (or $-a$)

(c) Same as for (a)

(d) Same as for (b)

(e) Same as for (a)

3.30. Closed paths are circles in z = constant plane and having centers on the z axis.

(a) 0 for $r<a$, $\frac{\mu_0 J_0}{2r}(r^2-a^2)\underset{\sim}{i}_\phi$ for $a<r<b$, $\frac{\mu_0 J_0}{2r}(b^2-a^2)\underset{\sim}{i}_\phi$ for $r>b$

(b) $\frac{\mu_0 J_0 r^{n+1}}{(n+2)a^n}\underset{\sim}{i}_\phi$ for $r<a$, $\frac{\mu_0 J_0 a^2}{(n+2)r}\underset{\sim}{i}_\phi$ for $r>a$

(c) $\frac{\mu_0 I r}{2\pi a^2}\underset{\sim}{i}_\phi$ for $r<a$, $\frac{\mu_0 I}{2\pi r}$ for $a<r<b$, $\frac{\mu_0 I(c^2-r^2)}{2\pi r(c^2-b^2)}\underset{\sim}{i}_\phi$ for $b<r<c$,

0 for $r>c$.

3.31. For answers, see page 533 of the text.

3.32. From considerations of symmetry, the magnetic field has only a ϕ component which is independent of ϕ. Also, the field outside the toroid is zero. Considering a circular path of radius r and lying inside the toroid and applying Ampere's circuital law, we obtain

$$2\pi r B_\phi = \mu_0[n(2\pi b)I]$$

$$\text{or,}\quad \underset{\sim}{B} = \begin{cases} \dfrac{\mu_0 n I b}{r}\, \underset{\sim}{i}_\phi & \text{for} \quad (r-b)^2 + z^2 < a^2 \\ 0 & \text{for} \quad (r-b)^2 + z^2 > a^2 \end{cases}$$

3.33. Total current flowing on the spherical surface is the same as the current in the filamentary wire. From symmetry considerations, the magnetic field has only a ϕ component which is independent of ϕ. Applying Ampere's circuital law to a circular path parallel to the xy plane and having center on the z axis, we obtain

$$2\pi r_c B_\phi = \mu_0 (\text{current enclosed}) = \begin{cases} \mu_0 I & \text{for } r_s > a \\ 0 & \text{for } r_s < a \end{cases}$$

which gives

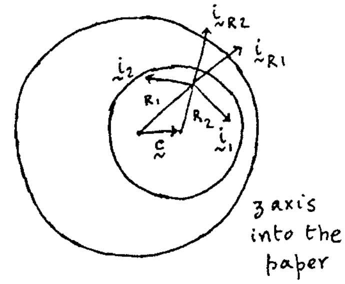
$$\underset{\sim}{B} = \begin{cases} \dfrac{\mu_0 I}{2\pi r_c}\, \underset{\sim}{i}_\phi & \text{for } r_s > a \\ 0 & \text{for } r_s < a \end{cases}$$

3.34. We make use of superposition to solve this problem by considering the given current distribution as the sum of two uniformly distributed axially directed current distributions, one of radius a and the other of radius b and such that the total current in the hole is zero. Then, we obtain the required magnetic flux density as

$$\underset{\sim}{B} = \frac{\mu_0 J_0 R_1}{2}\, \underset{\sim}{i}_1 + \frac{\mu_0 J_0 R_2}{2}\, \underset{\sim}{i}_2$$

$$= \frac{\mu_0 J_0}{2}\, \underset{\sim}{i}_z \times (R_1 \underset{\sim}{i}_{R1} - R_2 \underset{\sim}{i}_{R2})$$

$$= \frac{\mu_0 J_0}{2}\, \underset{\sim}{i}_z \times \underset{\sim}{c}$$



Thus the magnetic field inside the hole is uniform having magnitude $\dfrac{\mu_0 J_0 c}{2}$ and directed normal to the line joining the two centers.

3.35. All fields have only x components. Hence, Ampere's circuital law in differential form simplifies to $\dfrac{\partial B_x}{\partial y} = -\mu_0 J_z$. Verification consists of substituting B_x obtained in Problem 3.29. to check if J_z agrees with that given in Problem 3.29.

3.36. $\frac{1}{r}\frac{\partial}{\partial r}(r B_\phi) = \mu_0 J_z$. Procedure similar to Problem 3.35.

3.37. (a) $\underset{\sim}{J} = \frac{1}{\mu_0}\underset{\sim}{\nabla}\times\underset{\sim}{B} = -\frac{1}{\mu_0}\frac{\partial B_x}{\partial y}\underset{\sim}{i}_z$

(b) $\underset{\sim}{J} = \frac{1}{\mu_0}\underset{\sim}{\nabla}\times\underset{\sim}{B} = \frac{1}{\mu_0 r}\frac{\partial}{\partial r}(r B_\phi)\,\underset{\sim}{i}_z$

(c) $\underset{\sim}{J} = \frac{1}{\mu_0}\underset{\sim}{\nabla}\times\underset{\sim}{B} = \frac{1}{\mu_0 r}\left[\frac{\partial}{\partial r}(r B_\theta) - \frac{\partial B_r}{\partial \theta}\right]\underset{\sim}{i}_\phi$

For answers, see page 533 of the text.

3.38. We start with a volume current of density $\underset{\sim}{J} = J_x(x,z)\,\underset{\sim}{i}_x + J_y(x,z)\,\underset{\sim}{i}_y$ amp/m^2 between the plane surfaces $y = y_0 \pm \Delta y$. The integral of the volume current density with respect to y between the two plane surfaces is $[J_x(x,z)\,\underset{\sim}{i}_x + J_y(x,z)\,\underset{\sim}{i}_y]\, 2\Delta y$ amp/m. Let this quantity be $\underset{\sim}{J}_s = J_{sx}\,\underset{\sim}{i}_x + J_{sy}\,\underset{\sim}{i}_y$. If we now let Δy tend to zero increasing J_x and J_y such that $\underset{\sim}{J}_s$ remains constant, we obtain a surface current of density $\underset{\sim}{J}_s$ in the plane $y = y_0$. The quantity $\underset{\sim}{J}$ becomes a delta function $J_s\,\delta(y-y_0)$ since $\int_{y=y_0-}^{y_0+}\underset{\sim}{J}_s\,\delta(y-y_0)\,dy = \underset{\sim}{J}_s$.

Thus we obtain $\underset{\sim}{\nabla}\times\underset{\sim}{B} = \mu_0\underset{\sim}{J}_s\,\delta(y-y_0)$.

3.39. We start with a volume current of density $\underset{\sim}{J} = J_\phi(\phi,z)\,\underset{\sim}{i}_\phi + J_z(\phi,z)\,\underset{\sim}{i}_z$ amp/m^2 lying between the surfaces $r = r_0 \pm \Delta r$. The integral of the volume current density with respect to r between the two surfaces is $[J_\phi(\phi,z)\,\underset{\sim}{i}_\phi + J_z(\phi,z)\,\underset{\sim}{i}_z]\, 2\Delta r$ amp/m. Let this quantity be $\underset{\sim}{J}_s = J_{s\phi}\,\underset{\sim}{i}_\phi + J_{sz}\,\underset{\sim}{i}_z$. If now let Δr tend to zero increasing J_ϕ and J_z such that $\underset{\sim}{J}_s$ remains constant, we obtain a surface current of density $\underset{\sim}{J}_s$ on the surface $r = r_0$. The quantity $\underset{\sim}{J}$ becomes a delta function $\underset{\sim}{J}_s\,\delta(r-r_0)$ since $\int_{r=r_0-}^{r_0+} \underset{\sim}{J}_s\,\delta(r-r_0)\,dr = \underset{\sim}{J}_s$.

Thus we obtain $\underset{\sim}{\nabla}\times\underset{\sim}{B} = \mu_0\underset{\sim}{J}_s\,\delta(r-r_0)$.

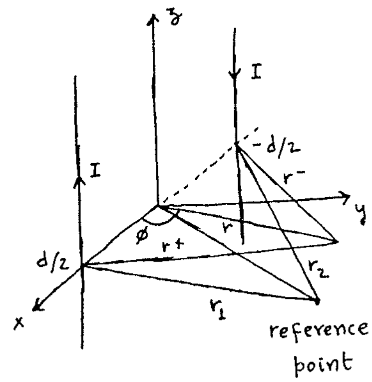
3.40. Solution follows in a manner similar to those of Problems 3.38. and 3.39. by starting with a z-directed volume current lying between the four surfaces $r = r_0 \pm \Delta r$ and $\phi = \phi_0 \pm \Delta\phi$. (See also Problem 2.33)

3.41. $\underset{\sim}{A} = \dfrac{\mu_0}{4\pi}\displaystyle\int_{z'=-a}^{a} \frac{I\,dz'}{\sqrt{r^2+(z-z')^2}}\,\underset{\sim}{i}_z = \frac{\mu_0 I}{4\pi}\ln\left[\frac{\sqrt{r^2+(z+a)^2}+(z+a)}{\sqrt{r^2+(z-a)^2}+(z-a)}\right]\underset{\sim}{i}_z$

$$\underset{\sim}{B} = \underset{\sim}{\nabla}\times\underset{\sim}{A} = -\frac{\partial A_z}{\partial r}\underset{\sim}{i}_\phi = \frac{\mu_0 I}{4\pi r}\left[\frac{z+a}{\sqrt{r^2+(z+a)^2}} - \frac{z-a}{\sqrt{r^2+(z-a)^2}}\right]\underset{\sim}{i}_\phi$$

$= \dfrac{\mu_0 I}{4\pi r}(\cos\alpha_1 - \cos\alpha_2)$ where α_1 and α_2 are as in Figure 3.7. of the text.

3.42. (a) $\underset{\sim}{A} = \left[-\dfrac{\mu_0 I}{2\pi}\ln\dfrac{r^+}{r_1} + \dfrac{\mu_0 I}{2\pi}\ln\dfrac{r^-}{r_2}\right]\underset{\sim}{i}_z$



(b) $\ln\dfrac{r^+}{r_1} = \ln\dfrac{\sqrt{r^2+(d/2)^2 - rd\cos\phi}}{r_1}$

$\approx \ln\dfrac{r}{r_0} - \dfrac{d}{2r}\cos\phi$ for $d\to 0$

keeping Id constant

$\ln\dfrac{r^-}{r_2} = \ln\dfrac{\sqrt{r^2+(d/2)^2 + rd\cos\phi}}{r_2}$

$\approx \ln\dfrac{r}{r_0} + \dfrac{d}{2r}\cos\phi$ for $d\to 0$ keeping Id constant.

$\therefore\ \underset{\sim}{A} = \dfrac{\mu_0 I d}{2\pi r}\cos\phi\,\underset{\sim}{i}_z$

(c) $\underset{\sim}{B} = \underset{\sim}{\nabla}\times\underset{\sim}{A} = \dfrac{\underset{\sim}{i}_r}{r}\dfrac{\partial A_z}{\partial\phi} - \underset{\sim}{i}_\phi\dfrac{\partial A_z}{\partial r} = \dfrac{\mu_0 I d}{2\pi r^2}(-\sin\phi\,\underset{\sim}{i}_r + \cos\phi\,\underset{\sim}{i}_\phi)$

which is the same as that found in Problem 3.21.

3.43. With reference to Figure 3.9. of the text,

$$\underset{\sim}{A} = \frac{\mu_0}{4\pi}\int_{\phi'=0}^{2\pi}\frac{Ia\,d\phi'\,\underset{\sim}{i}_\phi}{R}$$

$$= -\left[\frac{\mu_0}{4\pi}\int_{\phi'=0}^{2\pi}\frac{Ia\sin\phi'\,d\phi'}{R}\right]\underset{\sim}{i}_x + \left[\frac{\mu_0}{4\pi}\int_{\phi'=0}^{2\pi}\frac{Ia\cos\phi'\,d\phi'}{R}\right]\underset{\sim}{i}_y$$

The quantities $\displaystyle\int_{\phi'=0}^{2\pi}\frac{Ia\sin\phi'\,d\phi'}{R}$ and $\displaystyle\int_{\phi'=0}^{2\pi}\frac{Ia\cos\phi'\,d\phi'}{R}$ are $4\pi\epsilon_0$ times the electrostatic potentials due to ring charges of densities $I\sin\phi$ and $I\cos\phi$, respectively, occupying the position of the magnetic dipole. These potentials can be found for large values of r by using the method of Section 2.10. Thus, for example, for the ring charge of density $\rho_L = Ia\sin\phi$,

$$\int dQ = \int_{\phi=0}^{2\pi} (Ia\sin\phi)\, a\, d\phi = 0$$

$$\int dQ\, \underset{\sim}{r}' = \int_{\phi=0}^{2\pi} (Ia\sin\phi)(a\,d\phi)(a\cos\phi\, \underset{\sim}{i}_x + a\sin\phi\, \underset{\sim}{i}_y) = \pi I a^2\, \underset{\sim}{i}_y$$

$$[V]_{r\gg a} = \frac{\int dQ}{4\pi\epsilon_0 r} + \frac{\int dQ\, \underset{\sim}{r}'\cdot \underset{\sim}{r}}{4\pi\epsilon_0 r^3} + \cdots = \frac{\pi I a^2 \sin\theta \sin\phi}{4\pi\epsilon_0 r^2}$$

Similarly, finding $[V]_{r\gg a}$ for the ring charge of density $I a\cos\phi$, and substituting for the integrals, we obtain

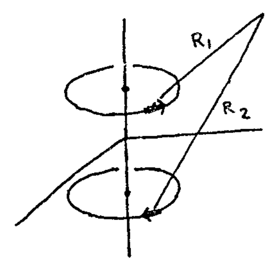
$$\underset{\sim}{A} = -\frac{\mu_0 \pi I a^2 \sin\theta \sin\phi}{4\pi r^2}\, \underset{\sim}{i}_x + \frac{\mu_0 \pi I a^2 \sin\theta\cos\phi}{4\pi r^2}\, \underset{\sim}{i}_y = \frac{\mu_0 m \sin\theta}{4\pi r^2}\, \underset{\sim}{i}_\phi$$

where $m = I\pi a^2$ is the magnitude of the dipole moment.

$$\underset{\sim}{B} = \underset{\sim}{\nabla} \times \underset{\sim}{A} = \frac{\underset{\sim}{i}_r}{r^2 \sin\theta} \frac{\partial}{\partial\theta}(r\sin\theta\, A_\phi) - \frac{\underset{\sim}{i}_\theta}{r\sin\theta}\frac{\partial}{\partial r}(r\sin\theta\, A_\phi)$$

$$= \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\, \underset{\sim}{i}_r + \sin\theta\, \underset{\sim}{i}_\theta)$$ which is the same as obtained in Example 3-4.

3.44. $$\underset{\sim}{A} = \frac{\mu_0}{4\pi}\left[\int_{\phi'=0}^{2\pi} \frac{I a\, d\phi'}{R_1}\, \underset{\sim}{i}_\phi - \int_{\phi'=0}^{2\pi} \frac{I a\, d\phi'}{R_2}\, \underset{\sim}{i}_\phi\right]$$



$$= -\frac{\mu_0}{4\pi}\int_{\phi'=0}^{2\pi}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) I a \sin\phi'\, d\phi'\, \underset{\sim}{i}_x$$

$$+ \frac{\mu_0}{4\pi}\int_{\phi'=0}^{2\pi}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) I a \cos\phi'\, d\phi'\, \underset{\sim}{i}_y$$

The quantities $\int_{\phi'=0}^{2\pi}\left(\frac{1}{R_1}-\frac{1}{R_2}\right) I a\sin\phi'\, d\phi'$ and $\int_{\phi'=0}^{2\pi}\left(\frac{1}{R_1}-\frac{1}{R_2}\right) I a\cos\phi'\, d\phi'$ are $4\pi\epsilon_0$ times the electrostatic potentials due to quadrupoles consisting of ring charges $\pm I\sin\phi$ and $\pm I\cos\phi$, respectively, and occupying the position of the magnetic quadrupole. These potentials can be found for large values of r by using the method of Section 2.10, which is also illustrated in Problem 3.43, noting however that we have to evaluate the terms $\int dQ[3(\underset{\sim}{r}'\cdot\underset{\sim}{r})^2 - r^2 r'^2]$ in this case. Evaluating these potentials and substituting for the integrals, we obtain

$$\underset{\sim}{A} = \frac{\mu_0 I}{4\pi r^3}\left(-6\pi a^2 d \sin\theta\sin\phi\cos\theta\, \underset{\sim}{i}_x + 6\pi a^2 d\sin\theta\cos\phi\cos\theta\, \underset{\sim}{i}_y\right)$$

$$= \frac{\mu_0 I}{4\pi r^3} 6\pi a^2 d \sin\theta \cos\theta \, \underset{\sim}{i}_\phi$$

$$\underset{\sim}{B} = \underset{\sim}{\nabla} \times \underset{\sim}{A} = \frac{\underset{\sim}{i}_r}{r^2 \sin\theta} \frac{\partial}{\partial\theta}(r\sin\theta \, A_\phi) - \frac{\underset{\sim}{i}_\theta}{r\sin\theta} \frac{\partial}{\partial r}(r\sin\theta \, A_\phi)$$

$$= \frac{3\mu_0 I a^2 d}{2r^4} \left[(3\cos^2\theta - 1)\, \underset{\sim}{i}_r + 2\sin\theta\cos\theta \, \underset{\sim}{i}_\theta \right]$$

which checks with the results for the special cases of Problem 3.23.

3.45. All vector potentials have only z components dependent only on y. From $\underset{\sim}{B} = \underset{\sim}{\nabla} \times \underset{\sim}{A}$, we then have $B_x = \frac{\partial A_z}{\partial y}$ or, $A_z = \int_{y=y_0}^{y} B_x \, dy$ where $y = y_0$ is taken to be the reference plane for zero potential. Substituting for B_x from Problem 3.29, we obtain the answers given on pages 533 and 534 of the text. The reference planes for zero potential are $y=0$ for all cases. Alternatively, the expressions for $\underset{\sim}{A}$ can be written by analogy with the expressions for V for analogous charge distributions.

3.46. All vector potentials have only z components dependent only on r. From $\underset{\sim}{B} = \underset{\sim}{\nabla} \times \underset{\sim}{A}$, we have $B_\phi = -\frac{\partial A_z}{\partial r}$ or $A_z = -\int_{r=r_0}^{r} B_\phi \, dr$ where $r = r_0$ is taken to be the reference surface for zero potential.

Substituting for B_ϕ from Problem 3.30, we obtain the following for $\underset{\sim}{A}$:

(a) 0 for $r < a$, $\frac{\mu_0 J_0}{2}\left(\frac{a^2 - r^2}{2} - a^2 \ln \frac{a}{r}\right) \underset{\sim}{i}_z$ for $a < r < b$,

$$\left[\frac{\mu_0 J_0}{2}\left(\frac{a^2 - b^2}{2} - a^2 \ln\frac{a}{b}\right) + \frac{\mu_0 J_0 (b^2 - a^2)}{2} \ln \frac{b}{r}\right] \underset{\sim}{i}_z \text{ for } r > b$$

(b) $\frac{\mu_0 J_0}{(n+2)^2 a^n}(a^{n+2} - r^{n+2})\, \underset{\sim}{i}_z$ for $r < a$, $\left[\frac{\mu_0 J_0 a^2}{n+2} \ln \frac{a}{r}\right] \underset{\sim}{i}_z$ for $r > a$

(c) $\frac{\mu_0 I}{2\pi}\left[\frac{c^2}{c^2 - b^2} \ln\frac{c}{b} + \ln\frac{b}{a} - \frac{r^2}{2a^2}\right] \underset{\sim}{i}_z$ for $r < a$,

$$\left[\frac{\mu_0 I}{2\pi(c^2 - b^2)}\left(c^2 \ln\frac{c}{b} + \frac{b^2 - c^2}{2}\right) + \frac{\mu_0 I}{2\pi} \ln\frac{b}{r}\right] \underset{\sim}{i}_z \text{ for } a < r < b,$$

$$\frac{\mu_0 I}{2\pi(c^2 - b^2)}\left(c^2 \ln\frac{c}{r} + \frac{r^2 - c^2}{2}\right) \underset{\sim}{i}_z \text{ for } b < r < c, \ 0 \text{ for } r > c.$$

3.47. (a) The magnetic vector potential is analogous to the electrostatic potential for the charge distribution given by

$$\rho_s = \begin{cases} \rho_{s0} & \text{for } y = a \\ -\rho_{s0} & \text{for } y = -a \end{cases}$$

for which V was found in Problem 2.53 (a).

(b) The magnetic vector potential is analogous to the electrostatic potential for the charge distribution given by

$$\rho_s = \begin{cases} \rho_{s0} & \text{for } r = a \\ -\rho_{s0}\, a/b & \text{for } r = b \end{cases}$$

for which V was found in Problem 2.53 (b).

For answers, see page 534 of the text. Alternatively, the answers can be obtained by using the expressions for $\underset{\sim}{B}$, found in Problem 3.31 (a) and (c), respectively, and following the procedures illustrated in Problems 3.45 and 3.46.

3.48. (a) $\underset{\sim}{m} = \frac{1}{2} \oint \underset{\sim}{r}' \times I\, d\underset{\sim}{l}'$

$$= \frac{1}{2} \int_{\phi'=0}^{\pi/2} (\cos\phi'\, \underset{\sim}{i}_x + \sin\phi'\, \underset{\sim}{i}_y) \times I\, d\phi' (-\sin\phi'\, \underset{\sim}{i}_x + \cos\phi'\, \underset{\sim}{i}_y)$$

$$+ \frac{1}{2} \int_{\theta'=\pi/2}^{0} (\cos\theta'\, \underset{\sim}{i}_z + \sin\theta'\, \underset{\sim}{i}_y) \times I\, d\theta' (-\sin\theta'\, \underset{\sim}{i}_z + \cos\theta'\, \underset{\sim}{i}_y)$$

$$+ \frac{1}{2} \int_{\theta'=0}^{\pi/2} (\cos\theta'\, \underset{\sim}{i}_z + \sin\theta'\, \underset{\sim}{i}_x) \times I\, d\theta' (-\sin\theta'\, \underset{\sim}{i}_z + \cos\theta'\, \underset{\sim}{i}_x)$$

$$= \frac{\pi}{4} I (\underset{\sim}{i}_x + \underset{\sim}{i}_y + \underset{\sim}{i}_z)$$

$$\underset{\sim}{A} = \frac{\mu_0}{4\pi r^3} \underset{\sim}{m} \times \underset{\sim}{r} = \frac{\mu_0}{4\pi r^3} \frac{\pi I}{4} (\underset{\sim}{i}_x + \underset{\sim}{i}_y + \underset{\sim}{i}_z) \times \underset{\sim}{r}$$

$$= \frac{\mu I}{16 r^2} [(\cos\theta - \sin\theta\sin\phi)\, \underset{\sim}{i}_x + (\sin\theta\cos\phi - \cos\theta)\, \underset{\sim}{i}_y$$
$$+ (\sin\theta\sin\phi - \sin\theta\cos\phi)\, \underset{\sim}{i}_z].$$

Alternatively, $\underset{\sim}{A} = \frac{\mu_0 I}{4\pi r}\left[\oint_{c'} d\underset{\sim}{l}' + \oint_{c'} \frac{\underset{\sim}{r}' \cdot \underset{\sim}{r}}{r^2}\, d\underset{\sim}{l}' + \cdots\right]$ can be used as in part (b) below.

(b) $\oint_c d\underset{\sim}{l}' \equiv 0$ because c' is a closd loop. $\oint_{c'} (\underset{\sim}{r}' \cdot \underset{\sim}{r})\, d\underset{\sim}{l}' = 0$ since the dipole moments due to the two halves of the loop are equal and opposite and hence cancel each other. Evaluating $\oint_{c'} [3(\underset{\sim}{r}' \cdot \underset{\sim}{r})^2 - r^2 r'^2]\, d\underset{\sim}{l}'$ and substituting in the expression for $\underset{\sim}{A}$, we obtain

$$\underset{\sim}{A} = \frac{\mu_0 I}{6\pi r^3} [(1 - 3\sin^2\theta\sin^2\phi - 3\sin^2\theta\sin\phi\cos\phi)\, \underset{\sim}{i}_x$$
$$- (1 - 3\sin^2\theta\cos^2\phi - 3\sin^2\theta\sin\phi\cos\phi)\, \underset{\sim}{i}_y].$$

3.49. $\mathbf{m} = \frac{1}{2}\oint_{C'} \mathbf{r}' \times I\,d\mathbf{l}'$

$$= \frac{1}{2}\int_{r'=0}^{a}\int_{\phi'=0}^{2\pi} (r'\cos\phi'\,\mathbf{i}_x + r'\sin\phi'\,\mathbf{i}_y) \times I n\,dr'(-r'\sin\phi'\,\mathbf{i}_x + r'\cos\phi'\,\mathbf{i}_y)\,d\phi'$$

$$= \pi I\left[\int_{r=0}^{a} n r^2\,dr\right]\mathbf{i}_z$$

For answers to (a), (b), and (c), see page 534 of the text.

3.50. We divide the spherical surface into several rings parallel to the xy plane and of infinitesimal width $a\,d\theta$ and write the dipole moment due to one such ring as $d\mathbf{m} = (nIa\,d\theta)(\pi a^2\sin^2\theta)\,\mathbf{i}_z$ which then gives $\mathbf{m} = \int_{\theta=0}^{\pi} d\mathbf{m} = \pi a^3 I \int_{\theta=0}^{\pi} n\sin^2\theta\,d\theta\;\mathbf{i}_z$.

(a) $\mathbf{m} = \frac{2}{3}\pi n_0 I a^3\,\mathbf{i}_z$, $\mathbf{A} = \dfrac{\mu_0 n_0 I a^3 \sin\theta}{6r^2}\,\mathbf{i}_\phi$

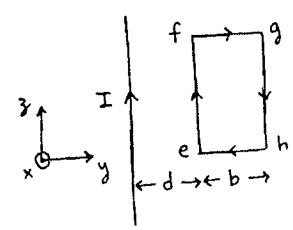
(b) $\mathbf{m} = 2\pi n_0 I a^3\,\mathbf{i}_z$, $\mathbf{A} = \dfrac{\mu_0 n_0 I a^3 \sin\theta}{2r^2}\,\mathbf{i}_\phi$

3.51. The spinning sphere is equivalent to a volume current of density $\mathbf{J} = \rho\mathbf{v} = \rho_0\omega_0 r\sin\theta\,\mathbf{i}_\phi$. We divide the spherical volume into a number of rings of infinitesimal areas of crosssection $(r\,d\theta)\,dr\,\mathbf{i}_\phi$ and write the dipole moment for one such ring as

$d\mathbf{m} = (\rho_0\omega_0 r\sin\theta)(r\,dr\,d\theta)(\pi r^2\sin^2\theta)\,\mathbf{i}_z$ which then gives

$$\mathbf{A} = \frac{\mu_0}{4\pi r^3}\left[\int_{r=0}^{a}\int_{\theta=0}^{\pi} d\mathbf{m}\right]\times\mathbf{r} = \frac{\mu_0\rho_0\omega_0 a^5\sin\theta}{15r^2}\,\mathbf{i}_\phi.$$

3.52. $\psi = \int_S \mathbf{B}\cdot d\mathbf{S} = \int_S \nabla\times\mathbf{A}\cdot d\mathbf{S} = \oint_C \mathbf{A}\cdot d\mathbf{l}$



From Example 3-11, the vector potential due to the infinitely long wire is $-\dfrac{\mu_0 I}{2\pi}\ln\dfrac{y}{y_0}\,\mathbf{i}_z$.

Hence

$$\oint_{efghe}\mathbf{A}\cdot d\mathbf{l} = \int_{z}^{z+a} -\frac{\mu_0 I}{2\pi}\ln\frac{d}{y_0}\,dz + 0 + \int_{z+a}^{z} -\frac{\mu_0 I}{2\pi}\ln\frac{d+b}{y_0}\,dz + 0$$

$$= \frac{\mu_0 I a}{2\pi}\ln\frac{d+b}{b}.$$

$$\int_S \mathbf{B}\cdot d\mathbf{S} = \int_{y=d}^{d+b}\int_{z=z}^{z+a} -\frac{\mu_0 I}{2\pi y}\,\mathbf{i}_x\cdot(-\mathbf{i}_x\,dy\,dz)$$

$$= \frac{\mu_0 I a}{2\pi}\ln\frac{d+b}{b}.$$

3.53. $\underset{\sim}{B} = \nabla \times \underset{\sim}{A} = -\frac{\partial A_z}{\partial y}\underset{\sim}{i}_x - \frac{\partial A_z}{\partial x}\underset{\sim}{i}_y$. The direction lines of $\underset{\sim}{B}$ are given by

$$\frac{dx}{\partial A_z/\partial y} = \frac{dy}{-\partial A_z/\partial x} = \frac{dz}{0}, \text{ or, } \frac{\partial A_z}{\partial x}dx + \frac{\partial A_z}{\partial y}dy = 0 \text{ and } z = 0,$$

or, $dA_z = 0$ and $z = 0$, or, $A_z =$ constant and $z =$ constant.

From Problem 3.42, $\underset{\sim}{A} = \left[\frac{\mu_0 I}{2\pi}\ln\frac{r^-}{r^+} + \text{constant}\right]\underset{\sim}{i}_z$. Hence, the direction lines of $\underset{\sim}{B}$ are given by $\frac{r^-}{r^+} =$ constant and $z =$ constant, or,

$$\frac{(x+d/2)^2 + y^2}{(x-d/2)^2 + y^2} = \text{constant, say, } c^2 \quad \text{and } z = \text{constant},$$

which represent circles as in Problem 3.22 (b).

3.54. If the divergence of a given field is zero, then it is realizable as a magnetic field. Hence, $\underset{\sim}{B}$, $\underset{\sim}{C}$, and $\underset{\sim}{D}$ can be realized as magnetic fields.

3.55. (a) Let the sheet current be z-directed with density J_{S0} and situated in the $y = 0$ plane. From symmetry considerations, we know that all components of $\underset{\sim}{B}$ are independent of x and z. Since a z-directed current does not produce a z component of $\underset{\sim}{B}$, we can say that $\underset{\sim}{B} = B_x(y)\,\underset{\sim}{i}_x + B_y(y)\,\underset{\sim}{i}_y$. Furthermore, B_y must be directed away from (or towards) the sheet and B_x must surround the sheet with each component having equal magnitudes at points equidistant on either side of the sheet. If we now consider a rectangular box enclosing and symmetrically situated about a portion of the current sheet and apply $\oint_S \underset{\sim}{B}\cdot d\underset{\sim}{S} = 0$, we get $B_y = 0$. To determine the only remaining component, B_x, we apply Ampere's circuital law to a rectangular path symmetrically situated about the current sheet and lying in a plane normal to it. This gives $2|B_x| = \mu_0 J_{S0}$. Thus

$$\underset{\sim}{B} = \begin{cases} \frac{\mu_0}{2}J_{S0}\,\underset{\sim}{i}_x & \text{for } y > 0 \\ -\frac{\mu_0}{2}J_{S0}\,\underset{\sim}{i}_x & \text{for } y < 0 \end{cases}$$

(b) Let the z axis be the axis of the cylinder and the current flow be in the z direction with density J_{S0}. From symmetry considerations and the fact that a z-directed current element does not produce a z component of $\underset{\sim}{B}$, we can write $\underset{\sim}{B} = B_r(r)\,\underset{\sim}{i}_r + B_\phi(r)\,\underset{\sim}{i}_\phi$. If we now consider a cylindrical box of length l and coaxial with the cylinder and apply $\oint \underset{\sim}{B}\cdot d\underset{\sim}{S} = 0$, we get $B_r = 0$. To determine the only remaining component, B_ϕ, we apply Ampere's circuital law to a circular path of radius r in the plane normal to the z axis and having its center on the z axis. This gives

$$2\pi r B_\phi = \begin{cases} \mu_0 2\pi a J_{S0} & \text{for } r > a \\ 0 & \text{for } r < a \end{cases}$$

$$\text{or,}\quad \underset{\sim}{B} = \begin{cases} \mu_0 J_{S0}\dfrac{a}{r}\,\underset{\sim}{i}_\phi & \text{for } r > a \\ 0 & \text{for } r < a \end{cases}$$

3.56. Let the loop be centered at the origin and in the xy plane with radius a. Since $B_\phi \equiv 0$ because of circular symmetry of the loop about the z axis, $\underset{\sim}{\nabla}\cdot\underset{\sim}{B} = 0$ reduces to $\dfrac{\partial B_r}{\partial r} + \dfrac{B_r}{r} + \dfrac{\partial B_z}{\partial z} = 0$. At a point on the z axis, $B_r = 0$ and $B_z = \dfrac{\mu_0 I a^2}{2(a^2+z^2)^{3/2}}$ from Problem 3.14. Thus

$$\left[\frac{\partial B_r}{\partial r}\right]_{z\text{ axis}} = -\left[\frac{\partial B_z}{\partial z}\right]_{z\text{ axis}} = \frac{3\mu_0 I a^2 z}{2(a^2+z^2)^{5/2}}.$$

3.57. See page 534 of the text for answers.

3.58.

Geometry		
Infinite line	Examples 2-4, 2-9	Example 3-3
Infinite sheet	Examples 2-5, 2-10	Examples 3-6, 3-9
Infinite slab	Problems 2.12, 2.23	Problems 3.26, 3.29
Infinitely long cylinder	Problem 2.13	Examples 3-7, 3-10
	Problem 2.24 (b)	Problem 3.30 (a)
Two-dimensional dipole	Problem 2.15	Problem 3.21
Infinite sheet-pair	Problem 2.26 (a)	Problem 3.31 (a)
Cylindrical surface(s)	Problem 2.26 (b), (c)	Problem 3.31 (b), (c)
Cylinder with hole	Problem 2.27	Problem 3.34

CHAPTER 4

4.1. Applying Lorentz force equation to three velocities $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, and the corresponding forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, we obtain

$$\mathbf{B} = \frac{(\mathbf{F}_2-\mathbf{F}_3)\times(\mathbf{F}_1-\mathbf{F}_2)}{q(\mathbf{F}_1-\mathbf{F}_2)\cdot(\mathbf{v}_2-\mathbf{v}_3)} = \frac{q\,\mathbf{i}_x \times q(-\mathbf{i}_x-\mathbf{i}_y)}{q^2(-\mathbf{i}_x-\mathbf{i}_y)\cdot(\mathbf{i}_y-\mathbf{i}_z)} = \mathbf{i}_z .$$

From $\mathbf{F}_1 = q[\mathbf{E}+\mathbf{v}_1\times\mathbf{B}]$, we then obtain $\mathbf{E} = \mathbf{i}_x + \mathbf{i}_y$.

4.2. From Lorentz force equation, we have $\mathbf{E}+\mathbf{v}_1\times\mathbf{B} = \mathbf{E}+\mathbf{v}_2\times\mathbf{B} = 0$, or, $(\mathbf{v}_2-\mathbf{v}_1)\times\mathbf{B} = 0$, or, $\mathbf{i}_z\times\mathbf{B} = 0$. Thus $\mathbf{B} = 0$ or $C\,\mathbf{i}_z$ where C is a constant. In either case, from $\mathbf{F}_3 = q(\mathbf{E}+\mathbf{v}_3\times\mathbf{B})$, we obtain $\mathbf{E} = \mathbf{i}_x + \mathbf{i}_y$. Then, from $\mathbf{v}_1\times\mathbf{B} = -\mathbf{E}$, we get $\mathbf{B} = \mathbf{i}_z$.

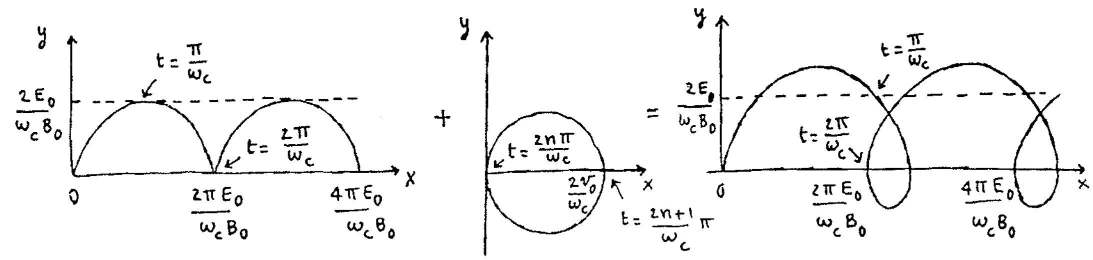
4.3. Method of solution is similar to that of Example 4-2, except that $v_y = v_0$ for $t=0$. Using this initial condition instead of $v_y = 0$ for $t=0$, we obtain

$$v_x = \frac{E_0}{B_0}(1-\cos\omega_c t) + v_0\sin\omega_c t \quad \text{and} \quad v_y = \frac{E_0}{B_0}\sin\omega_c t + v_0\cos\omega_c t$$

Proceeding further, we obtain

$$x = \frac{E_0}{\omega_c B_0}(\omega_c t - \sin\omega_c t) + \frac{v_0}{\omega_c}(1-\cos\omega_c t) \quad \text{and} \quad y = \frac{E_0}{\omega_c B_0}(1-\cos\omega_c t) + \frac{v_0}{\omega_c}\sin\omega_c t.$$

These equations represent a cycloid upon which is superimposed the curve given by $x = \frac{v_0}{\omega_c}(1-\cos\omega_c t)$ and $y = \frac{v_0}{\omega_c}\sin\omega_c t$. This curve is a circle with center at $\left(\frac{v_0}{\omega_c}, 0\right)$ and having radius $\frac{v_0}{\omega_c}$. Rough sketch of the path for small v_0 is shown below.



4.4. Method of solution is similar to that of Example 4-2 except that $v_x = v_0$ for $t=0$. Using this initial condition instead of $v_x = 0$ for $t=0$, we obtain

$$x = \frac{E_0}{\omega_c B_0}(\omega_c t - \sin \omega_c t) + \frac{v_0}{\omega_c} \sin \omega_c t$$

$$y = \frac{E_0}{\omega_c B_0}(1 - \cos \omega_c t) - \frac{v_0}{\omega_c}(1 - \cos \omega_c t)$$

These equations represent a cycloid upon which is superimposed the curve given by $x = \frac{v_0}{\omega_c} \sin \omega_c t$ and $y = -\frac{v_0}{\omega_c}(1 - \cos \omega_c t)$. This curve is a circle with center at $(0, -\frac{v_0}{\omega_c})$ and having radius $\frac{v_0}{\omega_c}$.

The paths of the test charge for different cases are as follows:

(a) Same as in Figure 4.2 of the text.

(b) $x = \frac{E_0}{\omega_c B_0}(\omega_c t - \frac{1}{2} \sin \omega_c t)$

$$y = \frac{E_0}{2\omega_c B_0}(1 - \cos \omega_c t)$$

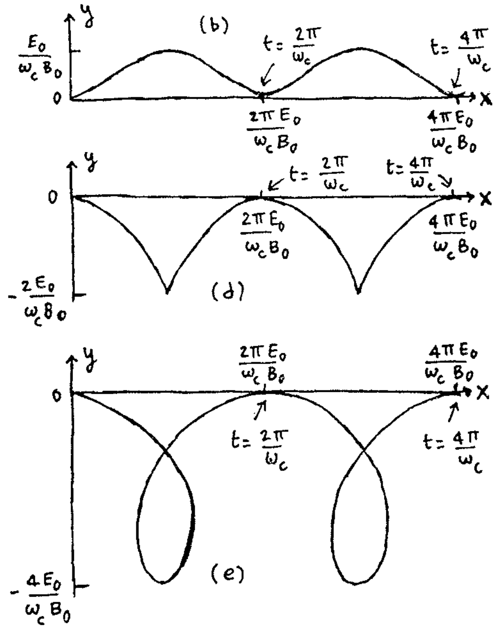
(c) $x = v_0 t$, $y = 0$, straight line path along the y axis

(d) $x = \frac{E_0}{\omega_c B_0}(\omega_c t + \sin \omega_c t)$

$$y = -\frac{E_0}{\omega_c B_0}(1 - \cos \omega_c t)$$

(e) $x = \frac{E_0}{\omega_c B_0}(\omega_c t + 2 \sin \omega_c t)$

$$y = -\frac{2E_0}{\omega_c B_0}(1 - \cos \omega_c t)$$



4.5. Writing the equations of motion of the test charge and eliminating v_y, we get $\frac{d^2 v_x}{dt^2} + \omega_c^2 v_x = \frac{qE_0}{m} \omega_c \cos \omega t$

The solution for v_x consists of two parts: the complementary function $C_1 \cos \omega_c t + C_2 \sin \omega_c t$ and the particular integral of the form $A \cos \omega t + B \sin \omega t$ for $\omega \neq \omega_c$. Substituting the particular in the differential equation and evaluating the coefficients, we obtain

$$v_x = \frac{qE_0}{m} \frac{\omega_c}{\omega_c^2 - \omega^2} \cos \omega t + C_1 \cos \omega_c t + C_2 \sin \omega_c t \quad \text{for } \omega \neq \omega_c.$$

From $\frac{dv_x}{dt} = \frac{qB_0}{m} v_y$, we then obtain

$$v_y = \frac{m}{qB_0}\left[-\frac{qE_0}{m} \frac{\omega_c \omega}{\omega_c^2 - \omega^2} \sin \omega t - \omega_c C_1 \sin \omega_c t + \omega_c C_2 \cos \omega_c t\right]$$

Substituting the initial conditions $v_x = v_y = 0$ for $t=0$ to evaluate c_1 and c_2 and then integrating, we get expressions for x and y, which upon substitution of the initial conditions $x = y = 0$ for $t=0$ give

$$x = \frac{qE_0}{m} \frac{\omega_c}{\omega_c^2-\omega^2}\left(\frac{\sin\omega t}{\omega} - \frac{\sin\omega_c t}{\omega_c}\right)$$

$$y = \frac{E_0}{B_0} \frac{\omega_c}{\omega_c^2-\omega^2}(\cos\omega t - \cos\omega_c t)$$

For $\omega \to 0$, these results agree with those obtained in Example 4-2.

For the case $\omega = \omega_c$, the procedure is similar except that the particular integral is of the form $At\cos\omega_c t + Bt\sin\omega_c t$ which gives solutions

$$x = \frac{qE_0}{2m\omega_c^2}(\sin\omega_c t - \omega_c t\cos\omega_c t) \quad \text{and} \quad y = \frac{E_0}{2B_0} t\sin\omega_c t.$$

4.6. Writing the equations of motion for the test charge and eliminating v_y, we get

$$\frac{d^2 v_x}{dt^2} + \omega_c^2 v_x = \frac{qE_0}{m}(\omega_c - \omega)\cos\omega t$$

which gives $v_x = \frac{qE_0}{m}\frac{1}{\omega_c+\omega}\cos\omega t + c_1\cos\omega_c t + c_2\sin\omega_c t$.

From $\frac{dv_x}{dt} = \frac{qB_0}{m}v_y - \frac{qE_0}{m}\sin\omega t$, we then obtain

$$v_y = \frac{m}{qB_0}\left[-\frac{qE_0}{m}\frac{\omega}{\omega_c+\omega}\sin\omega t - c_1\omega_c\sin\omega_c t + c_2\omega_c\cos\omega_c t + \frac{qE_0}{m}\sin\omega t\right]$$

Substituting the initial conditions $v_x = v_y = 0$ for $t=0$ to evaluate c_1 and c_2 and then integrating, we obtain expressions for x and y, which upon substitution of initial conditions $x=y=0$ for $t=0$ give

$$x = \frac{qE_0}{m(\omega_c+\omega)}\left[\frac{\sin\omega t}{\omega} - \frac{\sin\omega_c t}{\omega_c}\right]$$

$$y = \frac{qE_0}{m(\omega_c+\omega)}\left[-\frac{\cos\omega t}{\omega} - \frac{\cos\omega_c t}{\omega_c}\right] + \frac{E_0}{\omega B_0}$$

For $\omega \to 0$, these results agree with those of Example 4-2.

For $\omega = \omega_c$, $x = 0$ and $y = \frac{E_0}{\omega_c B_0}(1-\cos\omega_c t)$.

4.7. For loop at an arbitrary distance y from the z axis,

$$\psi = \int \mathbf{B}\cdot d\mathbf{S} = \int_y^{y+a}\int_z^{z+b}\left(-\frac{B_0}{y}\mathbf{i}_x\right)\cdot(-dy\,dz\,\mathbf{i}_x) = B_0 b \ln\frac{y+a}{y}$$

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d\psi}{dt} = -\frac{d}{dt}\left[B_0 b\ln\frac{y+a}{y}\right] = \frac{B_0 v_0 ab}{y(y+a)}.$$

4.8. $\psi = \int \mathbf{B}\cdot d\mathbf{s} = \int_y^{y+a}\int_z^{z+b} \left(-\frac{B_0}{y}\cos\omega t\, \mathbf{i}_x\right)\cdot(-dy\,dz\,\mathbf{i}_x) = B_0 b \cos\omega t \ln\frac{y+a}{y}$

$$\oint_c \mathbf{E}\cdot d\mathbf{l} = -\frac{d\psi}{dt} = \left(B_0 b \ln\frac{y+a}{a}\right)\omega \sin\omega t.$$

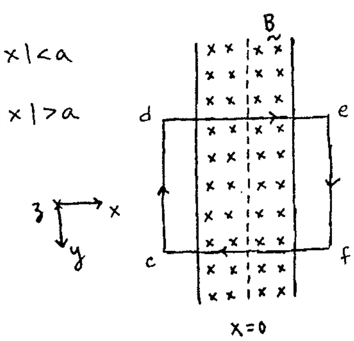
4.9. $\psi = \int \mathbf{B}\cdot d\mathbf{s} = B_0 b \ln\frac{y+a}{y}\cos\omega t$

$$\oint_c \mathbf{E}\cdot d\mathbf{l} = -\frac{d\psi}{dt} = B_0 b\omega \ln\frac{y+a}{y}\sin\omega t + \frac{B_0 v_0 ab}{y(y+a)}\cos\omega t.$$

4.10. (a) Since the induced electric field must surround the time varying magnetic field, it has only a y component. Also, from symmetry considerations, E_y is independent of y and z and must be an odd function of x. Considering then a rectangular path cdefc symmetrically situated about the x=0 plane as shown in the figure, we have

$$\oint_{cdefc} \mathbf{E}\cdot d\mathbf{l} = 2\,[E_y]_{cd}\,(cd)$$

$$\oint_{\text{area } cdef} \mathbf{B}\cdot d\mathbf{s} = \begin{cases} B_0 \sin\omega t \cdot 2x\,(cd) & \text{for } |x|<a \\ B_0 \sin\omega t \cdot 2a\,(cd) & \text{for } |x|>a \end{cases}$$



From Faraday's law, we then obtain

$$\mathbf{E} = \begin{cases} -\omega B_0 x \cos\omega t\, \mathbf{i}_y & \text{for } |x|<a \\ -\omega B_0 a \frac{|x|}{x}\cos\omega t\, \mathbf{i}_y & \text{for } |x|>a \end{cases}$$

(b) Method similar to that of Example 4-4.

$$\mathbf{E} = \begin{cases} 0 & \text{for } r<a \\ -\frac{r^2-a^2}{2r} B_0 \omega \cos\omega t\, \mathbf{i}_\phi & \text{for } a<r<b \\ -\frac{b^2-a^2}{2r} B_0 \omega\cos\omega t\, \mathbf{i}_\phi & \text{for } r>b \end{cases}$$

(c) Method similar to that of Example 4-4.

$$\mathbf{E} = \begin{cases} -\frac{2a^2r^2-r^4}{4a^2 r} B_0 \omega\cos\omega t\, \mathbf{i}_\phi & \text{for } r<a \\ -\frac{a^2}{4r} B_0 \omega \cos\omega t\, \mathbf{i}_\phi & \text{for } r>a \end{cases}$$

4.11. $\mathbf{E}' = \mathbf{E} + \mathbf{v}\times\mathbf{B} = 0 + \omega a\, \mathbf{i}_\phi \times B_0 \mathbf{i}_z = \omega a B_0 \mathbf{i}_r.$

4.12. From Problem 4.4, $\mathbf{v} = \left[\frac{E_0}{B_0} + \left(v_0 - \frac{E_0}{B_0}\right)\cos\omega_c t\right]\mathbf{i}_x - \left[\left(v_0 - \frac{E_0}{B_0}\right)\sin\omega_c t\right]\mathbf{i}_y$

For $v_0 = \frac{E_0}{B_0}$, $\mathbf{v} = \frac{E_0}{B_0}\mathbf{i}_x$, and $\mathbf{E}' = \mathbf{E} + \mathbf{v}\times\mathbf{B} = E_0 \mathbf{i}_y + \frac{E_0}{B_0}\mathbf{i}_x \times B_0 \mathbf{i}_z = 0.$

4.13. $-\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = -\int_y^{y+a}\int_z^{z+b} \frac{B_0}{y}\, \omega \sin \omega t\, \mathbf{i}_x \cdot (-dy\,dz\,\mathbf{i}_x) = B_0 b \omega \sin \omega t \ln \frac{y+a}{y}.$

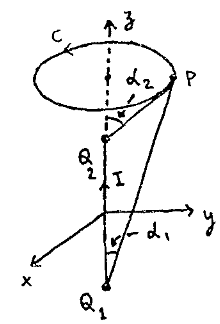
$$\oint_C \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l} = \int_z^{z+b} \frac{v_0 B_0}{y} \cos \omega t\, dz + 0 + \int_{z+b}^{z} \frac{v_0 B_0}{y+a} \cos \omega t\, dz + 0$$

$$= \frac{v_0 B_0 ab}{y(y+a)} \cos \omega t.$$

Adding the two, we obtain the same result as in Problem 4.9.

4.14. Verification consists of evaluating the curls of $\mathbf{E}$ obtained in Problem 4.10 and showing that they are the same as the corresponding expressions for $-\frac{\partial \mathbf{B}}{\partial t}$.

4.15. (a) From Example 3-3, the magnetic flux density at a point P on C is given by



$$\mathbf{B} = \frac{\mu_0 I}{4\pi a}(\cos \alpha_1 - \cos \alpha_2)\, \mathbf{i}_\phi$$

which then gives

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0 I}{2}\left[\frac{z+d}{\sqrt{a^2+(z+d)^2}} - \frac{z-d}{\sqrt{a^2+(z-d)^2}}\right]$$

(b) We consider the plane surface S bounded by C to apply the modified Ampere's circuital law to C.

We then have $[I_c]_S = 0$, and

$$[I_d]_S = \epsilon_0 \frac{d}{dt}\left[\frac{Q_1}{4\pi\epsilon_0}(\text{solid angle } \Omega_1 \text{ subtended by S at } Q_1) + \frac{Q_2}{4\pi\epsilon_0}(\text{solid angle } \Omega_2 \text{ subtended by S at } Q_2)\right]$$

substituting $\Omega_1 = 2\pi\left[1 - \frac{z+d}{\sqrt{a^2+(z+d)^2}}\right]$ and $\Omega_2 = 2\pi\left[1 - \frac{z-d}{\sqrt{a^2+(z-d)^2}}\right]$

and noting that $\frac{dQ_1}{dt} = -I$ and $\frac{dQ_2}{dt} = I$, we get

$$[I_d]_S = \frac{I}{2}\left[\frac{z+d}{\sqrt{a^2+(z+d)^2}} - \frac{z-d}{\sqrt{a^2+(z-d)^2}}\right]$$

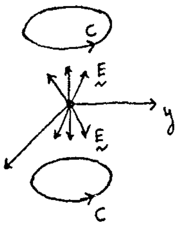
which gives the same result for $\oint_C \mathbf{B} \cdot d\mathbf{l}$ as in part (a).

4.16. (a) From Problem 3.25, the magnetic flux density due to the surface current distribution is $\mathbf{B} = \frac{\mu_0 I}{4\pi r}\left[\frac{z}{\sqrt{r^2+z^2}} - \frac{z}{|z|}\right]\mathbf{i}_\phi$ which gives

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0 I}{2}\left[\frac{z}{\sqrt{r^2+z^2}} - \frac{z}{|z|}\right].$$

(b) Considering the plane surface bounded by C, we have

$[I_c]_S = 0$, and



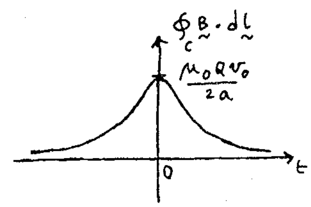
$$\int_S \underset{\sim}{E} \cdot d\underset{\sim}{S} = \begin{cases} \dfrac{Q}{4\pi\epsilon_0}\left[2\pi\left(1 - \dfrac{z}{\sqrt{r^2+z^2}}\right)\right] & \text{for C above the xy plane} \\ -\dfrac{Q}{4\pi\epsilon_0}\left[2\pi\left(1 + \dfrac{z}{\sqrt{r^2+z^2}}\right)\right] & \text{for C below the xy plane} \end{cases}$$

Substituting these in the modified Ampere's circuital law and noting that $\dfrac{dQ}{dt} = -I$, we get the same result as in part (a).

4.17. For path C outside the sphere, we consider a bowl shaped surface which does not cut any of the surface current on the spherical surface. For path C inside the sphere, we consider a plane surface. For answers, see page 534 of the text.

4.18. The position of the point charge on the z axis at any time t is $v_0 t$. Considering the plane surface S bounded by C, we have for a position z of the point charge, $[I_c]_S = 0$, and

$$[I_d]_S = \frac{d}{dt}\left\{-\frac{|z|}{z}\frac{Q}{4\pi}\left[2\pi\left(1 - \frac{|z|}{\sqrt{a^2+z^2}}\right)\right]\right\}$$



$$\oint_C \underset{\sim}{B} \cdot d\underset{\sim}{l} = -\frac{\mu_0 Q}{2}\frac{d}{dt}\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)$$

$$= \frac{\mu_0 Q}{2}\frac{a^2}{(a^2+z^2)^{3/2}}\frac{dz}{dt} = \frac{\mu_0 Q v_0}{2}\frac{a^2}{(a^2+v_0^2t^2)^{3/2}}$$

From symmetry considerations, $\oint_C \underset{\sim}{B} \cdot d\underset{\sim}{l} = 2\pi a B_\phi$

$$\therefore \underset{\sim}{B} = \frac{\mu_0 Q v_0}{4\pi}\frac{a}{(a^2+v_0^2t^2)^{3/2}}$$

4.19. Considering the plane surface S bounded by C, we have $[I_c]_S = I$, and

$[I_d]_S = \dfrac{d}{dt}\displaystyle\int_S \epsilon_0 \underset{\sim}{E} \cdot d\underset{\sim}{S} = \dfrac{d}{dt}$ (ϵ_0 times the electric field flux due to Q crossing S).

From symmetry considerations, this electric field flux is $\dfrac{Q}{8\epsilon_0}$. Thus

$[I_d]_S = \dfrac{d}{dt}\left(\dfrac{Q}{8}\right) = -\dfrac{1}{8}I$, and $\oint_C \underset{\sim}{B} \cdot d\underset{\sim}{l} = \dfrac{7}{8}\mu_0 I$.

4.20. We consider the surface S to be the plane surface bounded by C except for a slight upward bulge at point charge Q_2 to avoid the charge. Then

$[I_c]_S = 0$, $[I_d]_S = \dfrac{d}{dt}\left[\epsilon_0\left(\dfrac{Q_1}{8\epsilon_0} + \dfrac{Q_2}{2\epsilon_0}\right)\right] = \dfrac{1}{8}\dfrac{dQ_1}{dt} + \dfrac{1}{2}\dfrac{dQ_2}{dt} = \dfrac{1}{8}(-I) + \dfrac{1}{2}(I)$

$= \dfrac{3}{8}I$, and $\oint_C \underset{\sim}{B} \cdot d\underset{\sim}{l} = \dfrac{3}{8}\mu_0 I$.

4.21. We consider the surface S to be the plane surface bounded by C except for a slight bulge around Q_2 to the right of Q_2. Then, $[I_c]_S = 1$, and

$$[I_d]_S = \frac{d}{dt}\left[\frac{Q_1}{4\pi}2\pi\left(1-\frac{1}{\sqrt{2}}\right) + \frac{Q_2}{4\pi}2\pi - \frac{Q_3}{4\pi}2\pi\left(1-\frac{1}{\sqrt{2}}\right)\right]$$

$$= \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)\frac{dQ_1}{dt} + \frac{1}{2}\frac{dQ_2}{dt} - \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)\frac{dQ_3}{dt}$$

$$= \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)(-2) + \frac{1}{2}(1) - \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)(1) = 0.0606$$

$$\oint_C \mathbf{B}\cdot d\mathbf{l} = 1.0606\,\mu_0 .$$

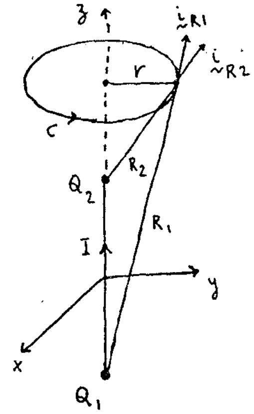
4.22. $$\nabla\times\mathbf{B} = \nabla\times\frac{\mu_0 I}{4\pi r}\left[\frac{z+d}{\sqrt{r^2+(z+d)^2}} - \frac{z-d}{\sqrt{r^2+(z-d)^2}}\right]\mathbf{i}_\phi$$

$$= -\frac{\mu_0 I}{4\pi[r^2+(z+d)^2]^{3/2}}[r\,\mathbf{i}_r + (z+d)\,\mathbf{i}_z]$$

$$+ \frac{\mu_0 I}{4\pi[r^2+(z-d)^2]^{3/2}}[r\,\mathbf{i}_r + (z-d)\,\mathbf{i}_z]$$

$$= \frac{\mu_0}{4\pi R_1^2}\left(\frac{dQ_1}{dt}\right)\mathbf{i}_{R1} + \frac{\mu_0}{4\pi R_2^2}\left(\frac{dQ_2}{dt}\right)\mathbf{i}_{R2}$$

$$= \mu_0\frac{\partial}{\partial t}\left[\frac{Q_1}{4\pi R_1^2}\mathbf{i}_{R1} + \frac{Q_2}{4\pi R_2^2}\mathbf{i}_{R2}\right] = \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}.$$



4.23. To evaluate $\nabla\times\mathbf{B}$ at a point on the xy plane, we have to first find $\nabla\times\mathbf{B}$ at an arbitrary point (r,ϕ,z) and then let $z\to 0$. Generalizing the result of Problem 4.18, we have $\mathbf{B}(r,\phi,z) = \frac{\mu_0 Q v_0}{4\pi}\frac{r}{[r^2+(v_0t-z)^2]^{3/2}}\mathbf{i}_\phi$ which gives

$$\nabla\times\mathbf{B} = \frac{\mu_0 Q v_0}{4\pi r}\left\{\frac{-3r^2(v_0t-z)\,\mathbf{i}_r + [-r^3+2r(v_0t-z)^2]\,\mathbf{i}_z}{[r^2+(v_0t-z)^2]^{5/2}}\right\}$$

$$[\nabla\times\mathbf{B}]_{z=0} = -\frac{\mu_0 Q}{4\pi}\frac{3rv_0^2t}{(r^2+v_0^2t^2)^{5/2}}\mathbf{i}_r + \frac{\mu_0 Q v_0}{4\pi}\frac{(-r^2+2v_0^2t^2)}{(r^2+v_0^2t^2)^{5/2}}\mathbf{i}_z$$

$$= \mu_0\frac{\partial}{\partial t}\left[\frac{Q}{4\pi(r^2+v_0^2t^2)^{3/2}}(r\,\mathbf{i}_r - v_0t\,\mathbf{i}_z)\right] = \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}.$$

4.24. $$\nabla\times\mathbf{B} = B_0\beta\cos\beta z\sin\omega t\,\mathbf{i}_y = \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}.$$

$$\therefore\ \mathbf{E} = -\frac{B_0\beta}{\omega\mu_0\epsilon_0}\cos\beta z\cos\omega t\,\mathbf{i}_y .$$

From $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$, we have $\frac{\partial E_y}{\partial z} = \frac{\partial B_x}{\partial t} = B_0\omega\sin\beta z\cos\omega t$.

$$\mathbf{E} = -\frac{B_0\omega}{\beta}\cos\beta z\cos\omega t\,\mathbf{i}_y .$$

Comparing the two expressions for $\mathbf{E}$, we get $\beta = \omega\sqrt{\mu_0\epsilon_0}$.

4.25. Work required $= (\sqrt{2}-1)\,\frac{1}{2}\Big[\frac{1}{4\pi\epsilon_0}\left(-2+\frac{3}{\sqrt{2}}+4\right) - \frac{2}{4\pi\epsilon_0}\left(1+3+\frac{4}{\sqrt{2}}\right)$

$$+\frac{3}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{2}}-2+4\right)+\frac{4}{4\pi\epsilon_0}\left(1-\frac{2}{\sqrt{2}}+3\right)\Big] = \frac{0.1471}{\epsilon_0}\ \text{N-m}.$$

4.26. (a) Using the result for V from Problem 2.52 (a), we have

$$W_e = \frac{1}{2}\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\rho_0\,\frac{\rho_0}{6\epsilon_0 r}(3b^2r - r^3 - 2a^3)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \frac{2\pi\rho_0^2}{15\epsilon_0}(2b^5+3a^5-5a^3b^2)\ \text{N-m}$$

(b) Using the result for V from Problem 2.52(b), we have

$$W_e = \frac{1}{2}\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\left(\rho_0\frac{r}{a}\right)\left[\frac{\rho_0}{12\epsilon_0 a}(4a^3-r^3)\right]r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \frac{\pi\rho_0^2 a^5}{7\epsilon_0}\ \text{N-m}.$$

4.27. Verification consists of evaluating $\int_V \frac{1}{2}\epsilon_0 E^2\,dv$ using the expressions for $\mathbf{E}$ obtained in Problem 2.25.

4.28. From Example 4-11, the potential energy associated with each charge is $\frac{4\pi\rho_0^2a^5}{15\epsilon_0}$. Hence, the potential energy associated with the two spherical charges is $\frac{8\pi\rho_0^2a^5}{15\epsilon_0}$.

(a) Radius of the new charge distribution $= 2^{1/3}a$

Potential energy $= \frac{4\pi\rho_0^2(2^{1/3}a)^5}{15\epsilon_0} = 3.175\,\frac{4\pi\rho_0^2a^5}{15\epsilon_0}$.

Work required $= 4.7\,\pi\rho_0^2a^5/15\epsilon_0$.

(b) charge density of the new charge distribution $= 2\rho_0$.

Potential energy $= \frac{4\pi(2\rho_0)^2a^5}{15\epsilon_0} = \frac{16\pi\rho_0^2a^5}{15\epsilon_0}$.

Work required $= 8\pi\rho_0^2a^5/15\epsilon_0$.

4.29. $W_e = \int_{vol}\frac{1}{2}\epsilon_0\,\mathbf{E}\cdot\mathbf{E}\,dv = \int_{vol}\frac{1}{2}\epsilon_0(\mathbf{E}_1+\mathbf{E}_2)\cdot(\mathbf{E}_1+\mathbf{E}_2)\,dv$

$$= \int_{vol}\left(\frac{1}{2}\epsilon_0E_1^2+\frac{1}{2}\epsilon_0E_2^2+\epsilon_0\mathbf{E}_1\cdot\mathbf{E}_2\right)dv.$$

4.30. (a) From Problem 2.53 (c), $V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$ for $r<a$, $\frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r}-\frac{1}{b}\right)$ for $a<r<b$, and 0 for $r>b$. Hence,

$$W_e = \frac{1}{2}\int\rho V\,dv = \frac{1}{2}\int_{r=a-}^{a+}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{Q}{4\pi a^2}\delta(r-a)\,\frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \frac{Q^2(b-a)}{8\pi\epsilon_0 ab}.$$

(b) From Problem 2.26(e), $\underset{\sim}{E} = 0$ for $r<a$, $\dfrac{Q}{4\pi\epsilon_0 r^2}\underset{\sim}{i}_r$ for $a<r<b$, and 0 for $r>b$.

$$\therefore W_e = \int \frac{1}{2}\epsilon_0 E^2 dv = \int_{r=a}^{b}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{Q^2}{16\pi^2\epsilon_0^2 r^4} r^2 \sin\theta \, dr\, d\theta\, d\phi = \frac{Q^2(b-a)}{8\pi\epsilon_0 ab}.$$

(c) Substituting $\underset{\sim}{E}_1 = \dfrac{Q}{4\pi\epsilon_0 r^2}\underset{\sim}{i}_r$ for $r>a$, 0 for $r<a$, and

$\underset{\sim}{E}_2 = -\dfrac{Q}{4\pi\epsilon_0 r^2}\underset{\sim}{i}_r$ for $r>b$, 0 for $r<b$ in the expression for W_e given

in Problem 4.29, we obtain the same result as in parts (a) and (b).

4.31. (a) Using the expression for $\underset{\sim}{A}$ obtained in Problem 3.46(c) and evaluating the integral, we obtain the result given on page 535 of the text.

$\underset{\sim}{A}$ can be obtained alternatively by using the method outlined in (b) below.

(b) From analogy with the electrostatic potential due to the surface charge of density $\rho_S = \rho_{S0}$ for $r=a$, the magnetic vector potential due to the surface current distribution $\underset{\sim}{J}_S = J_{S0}\,\underset{\sim}{i}_z$ for $r=a$ is given by

$$\underset{\sim}{A} = \begin{cases} \mu_0 J_{S0}\, a \ln \frac{r}{a}\, \underset{\sim}{i}_z & \text{for } r<a \\ \mu_0 J_{S0}\, a \ln \frac{r_0}{r}\, \underset{\sim}{i}_z & \text{for } r>a \end{cases}$$

where $r=r_0$ is the reference surface for zero potential. Using this result and choosing $r=b$, we obtain the vector potential in the region $r<a$ for the given current distribution as

$$\underset{\sim}{A} = \left[\int_{r'=0}^{r} \mu_0 J_0\left(\frac{r'}{a}\right) dr'\, r' \ln\frac{b}{r} + \int_{r'=r}^{a} \mu_0 J_0\left(\frac{r'}{a}\right) dr' \ln\frac{b}{r'}\right]\underset{\sim}{i}_z$$

$$= -\frac{\mu_0 J_0}{a}\left(\frac{a^3}{3}\ln\frac{a}{b} - \frac{a^3-r^3}{9}\right)\underset{\sim}{i}_z$$

Substituting this result in the integral for W_m, we get

$$W_m = \frac{1}{2}\int_{r=0}^{a}\int_{\phi=0}^{2\pi}\int_{z=0}^{1} J_0 \frac{r}{a}\left\{-\frac{\mu_0 J_0}{a}\left(\frac{a^3}{3}\ln\frac{a}{b} - \frac{a^3-r^3}{9}\right)\right\} r\, dr\, d\phi\, dz$$

$$= \frac{\pi\mu_0 J_0^2}{9}\left(a^4 \ln\frac{b}{a} + \frac{a^4}{6}\right)$$

4.32. (a) Using the expression for $\underset{\sim}{A}$ from Problem 3.47(a), we obtain

$$W_m = \frac{1}{2}\int_{z=0}^{1}\int_{x=0}^{1}(J_{S0}\,\underset{\sim}{i}_z)\cdot(\mu_0 J_{S0}\, a\, \underset{\sim}{i}_z)\, dx\, dz$$

$$+ \frac{1}{2}\int_{z=0}^{1}\int_{x=0}^{1}(-J_{S0}\,\underset{\sim}{i}_z)\cdot(-\mu_0 J_{S0}\, a\, \underset{\sim}{i}_z)\, dx\, dz$$

$$= \mu_0 J_{S0}\, a^2.$$

(b) Using the expression for $\underset{\sim}{A}$ from Problem 3.45 (d), we obtain

$$w_m = \frac{1}{2}\int_{y=-a}^{a}\int_{x=0}^{1}\int_{z=0}^{1}(y\,\underset{\sim}{i}_z)\cdot\left(\mu_0\,\frac{3a^2y-y^3}{6}\,\underset{\sim}{i}_z\right)dx\,dy\,dz = \frac{2\mu_0}{15}.$$

4.33. Magnetic field for the current distribution of Problem 4.31 (a) is given in Problem 3.30 (c).

Magnetic field for the current distribution of Problem 4.31 (b) obtained by using Ampere's circuital law is given by

$\underset{\sim}{B} = \dfrac{\mu_0 J_0 r^2}{3a}\,\underset{\sim}{i}_\phi$ for $r<a$, $\dfrac{\mu_0 J_0 a^2}{3r}\,\underset{\sim}{i}_\phi$ for $a<r<b$, and 0 for $r>b$.

Magnetic field for the current distribution of Problem 4.32 (a) is given in Problem 3.31 (a).

Magnetic field for the current distribution of Problem 4.32 (b) is given in Problem 3.29 (d).

Using these expressions for $\underset{\sim}{B}$ and evaluating $\int \frac{1}{2}\frac{B^2}{\mu_0}$, we get the same results as in Problems 4.31 and 4.32.

4.34. $$W_m = \int_{vol}\frac{1}{2\mu_0}\underset{\sim}{B}\cdot\underset{\sim}{B}\,dv = \int_{vol}\frac{1}{2\mu_0}(\underset{\sim}{B}_1+\underset{\sim}{B}_2)\cdot(\underset{\sim}{B}_1+\underset{\sim}{B}_2)\,dv$$

$$= \int_{vol}\left(\frac{B_1^2}{2\mu_0}+\frac{B_2^2}{2\mu_0}+\frac{\underset{\sim}{B}_1\cdot\underset{\sim}{B}_2}{\mu_0}\right)dv$$

4.35. (a) By using superposition in conjunction with the vector potential for the current distribution of Problem 3.47 (b) and considering $r=c$ as the reference surface for zero potential, we obtain the required vector potentials on the current carrying surfaces as

$\underset{\sim}{A} = \mu_0\left(I_1\ln\frac{c}{a}+I_2\ln\frac{c}{b}\right)\underset{\sim}{i}_z$ for $r=a$, $\mu_0(I_1+I_2)\ln\frac{c}{b}$ for $r=b$, and 0 for $r=c$. We then have

$$w_m = \frac{1}{2}\int_{\phi=0}^{2\pi}\int_{z=0}^{1}\left(\frac{I_1}{a}\underset{\sim}{i}_z\right)\cdot\left(\mu_0 I_1\ln\frac{c}{a}+\mu_0 I_2\ln\frac{c}{b}\right)\underset{\sim}{i}_z\,a\,d\phi\,dz$$

$$+\frac{1}{2}\int_{\phi=0}^{2\pi}\int_{z=0}^{1}\frac{I_2}{b}\underset{\sim}{i}_z\cdot\left[\mu_0(I_1+I_2)\ln\frac{c}{b}\right]\underset{\sim}{i}_z\,b\,d\phi\,dz$$

$$= \pi\mu_0\left(I_1^2\ln\frac{c}{a}+2I_1I_2\ln\frac{c}{b}+I_2^2\ln\frac{c}{b}\right).$$

(b) $\underset{\sim}{B}=0$ for $r<a$, $\mu_0\dfrac{I_1}{r}\underset{\sim}{i}_\phi$ for $a<r<b$, $\dfrac{\mu_0(I_1+I_2)}{r}\underset{\sim}{i}_\phi$ for $b<r<c$, 0 for $r>c$.

Evaluating $\int\frac{1}{2}\frac{B^2}{\mu_0}dv$, we get the same result as in part (a).

(c) Substituting $\underset{\sim}{B}_1 = \dfrac{\mu_0 I_1}{r}\underset{\sim}{i}_\phi$ for $a<r<c$, 0 otherwise and

$B_2 = \frac{\mu_0 I_2}{r} i_\phi$ for $b<r<c$, 0 elsewhere in the expression for w_m given in Problem 4.34, we obtain the same result as in parts (a) and (b).

4.36. $\frac{\partial B}{\partial t} = -\nabla \times E = 50\beta \cos(\omega t + \beta z)\, i_x - 100\beta \sin(\omega t - \beta z)\, i_y$

$$B = \frac{50\beta}{\omega} \sin(\omega t + \beta z)\, i_x + \frac{100\beta}{\omega} \cos(\omega t - \beta z)\, i_y.$$

$$P = \sqrt{\frac{\epsilon_0}{\mu_0}} \left[10{,}000 \cos^2(\omega t - \beta z) - 2{,}500 \sin^2(\omega t - \beta z)\right] i_z.$$

4.37. $P = E \times \frac{B}{\mu_0} = \frac{V_0 I_0}{8\pi r^2 \ln \frac{b}{a}} \sin 2\beta z \sin 2\omega t\, i_z.$

Required power $= \int_S P \cdot dS$ where S is the surface bounding the volume

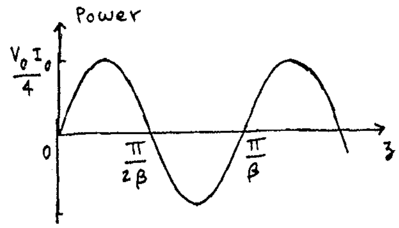
$$= \int_{r=a}^{b} \int_{\phi=0}^{2\pi} [P]_{z=0} \cdot [-r\,dr\,d\phi\, i_z] + \int_{r=a}^{b} \int_{\phi=0}^{2\pi} [P]_{z=z} \cdot [r\,dr\,d\phi\, i_z]$$

$$+ \int_{\substack{\text{cylindrical}\\ \text{surfaces}}} P \cdot dS$$

$$= 0 + \frac{V_0 I_0}{4 \ln \frac{b}{a}} \sin 2\beta z \sin 2\omega t \int_a^b \frac{dr}{r} + 0$$

$$= \frac{V_0 I_0}{4} \sin 2\beta z \sin 2\omega t$$

$$\left[\int_S P \cdot dS\right]_{\omega t = \frac{\pi}{4}} = \frac{V_0 I_0}{4} \sin 2\beta z$$



4.38. Applying Ampere's circuital law to the current distribution, we get

$$B = \frac{\mu_0 J_0 r}{2} i_\phi \text{ for } r<a, \text{ and } \frac{\mu_0 J_0 a^2}{2r} i_\phi \text{ for } r>a.$$

For $r<a$, $E \times B = E_0\, i_z \times \frac{\mu_0 J_0 r}{2} i_\phi = -\frac{\mu_0 E_0 J_0 r}{2} i_r.$

$$\oint_S E \times \frac{B}{\mu_0} \cdot dS = \int_{\substack{\text{side}\\ \text{surfaces}}} E \times \frac{B}{\mu_0} \cdot dS + \int_{\substack{\text{cylindrical}\\ \text{surface}}} E \times \frac{B}{\mu_0} \cdot dS$$

$$= 0 + \int_{z=0}^{l} \int_{\phi=0}^{2\pi} \left(-\frac{E_0 J_0 r}{2} i_r\right) \cdot (r\,d\phi\,dz\, i_r) = -\pi E_0 J_0 r^2 l.$$

The minus sign indicates that the power flow is into the volume. Now, the power expended by the field in constituting the flow of current is

$$\int_V E \cdot J\, dv = \int_{r=0}^{r} \int_{\phi=0}^{2\pi} \int_{z=0}^{l} (E_0\, i_z \cdot J_0\, i_z)\, r\,dr\,d\phi\,dz = \pi E_0 J_0 r^2 l$$

which is the same as that obtained by surface integration of the Poynting vector.

4.39. Method similar to that of Example 4-13. For answer, see page 535 of the text.

4.40. (a) Assuming solution of the form $V = V_m \cos(500t + \theta)$ and substituting into the differential equation and solving for V_m and θ, we get

$$V = \frac{10}{\sqrt{2}} \cos\left(500t - \frac{7\pi}{12}\right).$$

(b) $2 \times 10^{-3}\,(j500)\,\bar{V} + \bar{V} = 10\, e^{-j\frac{\pi}{3}}$

$$\bar{V} = \frac{10}{\sqrt{2}} e^{-j\frac{7\pi}{12}}, \quad V = \frac{10}{\sqrt{2}} \cos\left(500t - \frac{7\pi}{12}\right).$$

4.41. Solution similar to that of Problem 4.40. For answer, see page 535 of the text.

4.42. At point A, $\underset{\sim}{B} = -\dfrac{I_m}{2\pi a} \cos \omega t\; \underset{\sim}{i}_y + \dfrac{I_m}{2\pi a} \cos(\omega t + 90^\circ)\; \underset{\sim}{i}_y$

$$= \frac{I_m}{\sqrt{2}\,\pi a} \cos(\omega t + 135^\circ)\; \underset{\sim}{i}_y.$$

The field is linearly polarized in the y direction.

At point B, $\underset{\sim}{B} = \dfrac{I_1}{4\pi a}(\underset{\sim}{i}_x - \underset{\sim}{i}_y) + \dfrac{I_2}{4\pi a}(-\underset{\sim}{i}_x - \underset{\sim}{i}_y)$

$$B_x = \frac{I_m}{4\pi a}\left[\cos \omega t - \cos(\omega t + 90^\circ)\right] = \frac{I_m}{2\sqrt{2}\,\pi a} \cos(\omega t - 45^\circ)$$

$$B_y = \frac{I_m}{4\pi a}\left[-\cos \omega t - \cos(\omega t + 90^\circ)\right] = \frac{I_m}{2\sqrt{2}\,\pi a} \sin(\omega t - 45^\circ)$$

B_x and B_y are equal in magnitude and differ in phase by $\frac{\pi}{2}$.

Hence the field is circularly polarized.

At point C, $\underset{\sim}{B} = \dfrac{I_1}{10\pi a}(\underset{\sim}{i}_x - 2\,\underset{\sim}{i}_y) + \dfrac{I_2}{2\pi a}\,\underset{\sim}{i}_x$

$$B_x = \frac{I_m}{10\pi a} \cos \omega t + \frac{I_m}{2\pi a} \cos(\omega t + 90^\circ) = \frac{0.51}{\pi a} I_m \cos(\omega t + 78^\circ 41')$$

$$B_y = -\frac{I_m}{5\pi a} \cos \omega t = -\frac{0.2}{\pi a} I_m \cos \omega t.$$

B_x and B_y are unequal in magnitude and differ in phase by $78^\circ 41'$.

The field is elliptically polarized.

4.43. Method similar to that of Example 4-14. For answers, see page 535 of the text.

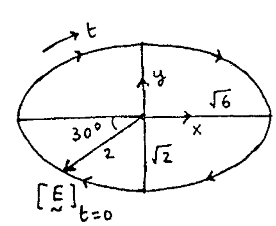
4.44. $\bar{E}_x = -(1 \cos 30^\circ)\angle 0^\circ + (1 \cos 30^\circ)\angle 90^\circ + (1 \cos 30^\circ)\angle 180^\circ - (1 \cos 30^\circ)\angle 270^\circ = \sqrt{6}\angle 135^\circ.$

$\bar{E}_y = -(1 \sin 30^\circ)\angle 0^\circ - (1 \sin 30^\circ)\angle 90^\circ + (1 \sin 30^\circ)\angle 180^\circ + (1 \sin 30^\circ)\angle 270^\circ = \sqrt{2}\angle 225^\circ.$

$E_x(t) = \sqrt{6} \cos(\omega t + 135^\circ)$

$E_y(t) = \sqrt{2} \cos(\omega t + 225^\circ) = -\sqrt{2} \sin(\omega t + 135^\circ).$

$\left(\dfrac{E_x}{\sqrt{6}}\right)^2 + \left(\dfrac{E_y}{\sqrt{2}}\right)^2 = 1.$ The field is elliptically polarized as shown in the figure.



4.45. (a) The phase angle is $-0.04\pi(\sqrt{3}x - 2y + 3z)$. Hence, surfaces of constant phase are planes $\sqrt{3}\,x - 2y - 3z = \text{constant}$.

(b) Since all field components have phase differences of $0°$ or $180°$, the field is linearly polarized. A normal vector to the constant phase surfaces is $\sqrt{3}\,\mathbf{i}_x - 2\,\mathbf{i}_y - 3\,\mathbf{i}_z$. Since $(-\mathbf{i}_x - 2\sqrt{3}\,\mathbf{i}_y + \sqrt{3}\,\mathbf{i}_z)\cdot(\sqrt{3}\,\mathbf{i}_x - 2\mathbf{i}_y - 3\mathbf{i}_z) = 0$, the field vector is linearly polarized in the planes of constant phase.

(c) See page 535 of the text for the answer.

4.46. (a) $\sqrt{3}\,x + z = \text{constant}$.

(b) $\bar{\mathbf{E}} = (-2\,\mathbf{i}_y + j2\,\mathbf{i}_{xz})\, e^{-j0.05\pi(\sqrt{3}\,x + z)}$

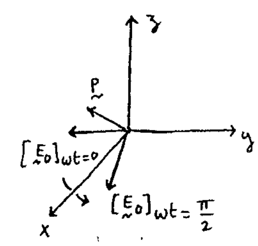
where $\mathbf{i}_{xz} = -\frac{1}{2}\,\mathbf{i}_x + \frac{\sqrt{3}}{2}\,\mathbf{i}_z$. Since $\mathbf{E}$ has two components perpendicular to each other and having equal magnitudes but differing in phase by $\frac{\pi}{2}$, it is circularly polarized. Normal vector to the constant phase surfaces is $\sqrt{3}\,\mathbf{i}_x + \mathbf{i}_z$. Since $(-j1\,\mathbf{i}_x - 2\,\mathbf{i}_y + j\sqrt{3}\,\mathbf{i}_z)\cdot(\sqrt{3}\,\mathbf{i}_x + \mathbf{i}_z) = 0$, the field vector is circularly polarized in the planes of constant phase.

(c) $\bar{\mathbf{B}} = \frac{0.1\pi}{\omega}(1\,\mathbf{i}_x - j2\,\mathbf{i}_y - \sqrt{3}\,\mathbf{i}_z)\, e^{-j0.05\pi(\sqrt{3}\,x + z)}$

$\mathbf{E}_0 = 1 \sin\omega t\, \mathbf{i}_x - 2\cos\omega t\, \mathbf{i}_y - \sqrt{3}\sin\omega t\, \mathbf{i}_z$

$\mathbf{B}_0 = \frac{0.1}{\pi}(1\cos\omega t\, \mathbf{i}_x + 2\sin\omega t\, \mathbf{i}_y - \sqrt{3}\cos\omega t\, \mathbf{i}_z)$

$\mathbf{P} = \frac{\mathbf{E}_0 \times \mathbf{B}_0}{\mu_0} = \frac{0.2\pi}{\mu_0}(\sqrt{3}\,\mathbf{i}_x + 1\,\mathbf{i}_z)$.



The field vector is right circularly polarized.

4.47. Method similar to that of Problem 4.46 if we note that

$$\bar{\mathbf{E}} = \bar{\mathbf{E}}_1 + \bar{\mathbf{E}}_2 = (-\sqrt{3}\,\mathbf{i}_x + \mathbf{i}_y)\, e^{-j0.02\pi(\sqrt{3}\,x + 3y + 2z)} + \left(-j\tfrac{1}{2}\,\mathbf{i}_x - j\tfrac{\sqrt{3}}{2}\,\mathbf{i}_y + j\sqrt{3}\,\mathbf{i}_z\right) e^{-j0.02\pi(\sqrt{3}\,x + 3y + 2z)}$$

(a) $\sqrt{3}\,x + 3y + 2z = \text{constant}$

(b) $\mathbf{E}_1$ and $\mathbf{E}_2$ are perpendicular to each other, equal in magnitude, and differ in phase by $\frac{\pi}{2}$. $\mathbf{E}_1$ and $\mathbf{E}_2$ are also perpendicular to $\sqrt{3}\,\mathbf{i}_x + 3\mathbf{i}_y + 2\mathbf{i}_z$. Hence the field is circularly polarized in the planes of constant phase.

(c) See page 535 of the text for answer.

4.48. $\bar{\mathbf{E}} = E_0 e^{j\theta}\,\mathbf{i}_x = \left(\frac{E_0}{2} e^{j\theta}\,\mathbf{i}_x + j\frac{E_0}{2} e^{j\theta}\,\mathbf{i}_y\right) + \left(\frac{E_0}{2} e^{j\theta}\,\mathbf{i}_x - j\frac{E_0}{2} e^{j\theta}\,\mathbf{i}_y\right)$

Thus a linearly polarized vector can be expressed as the superposition of two circularly polarized vectors of equal magnitude and opposite senses of rotation.

$$\bar{\mathbf{E}} = E_1 e^{j\theta}\,\mathbf{i}_x + E_2 e^{j\phi}\,\mathbf{i}_y$$

$$= \left(\frac{E_1 e^{j\theta} - jE_2 e^{j\phi}}{2}\,\mathbf{i}_x + j\,\frac{E_1 e^{j\theta} - jE_2 e^{j\phi}}{2}\,\mathbf{i}_y \right)$$

$$+ \left(\frac{E_1 e^{j\theta} + jE_2 e^{j\phi}}{2}\,\mathbf{i}_x - j\,\frac{E_1 e^{j\theta} + jE_2 e^{j\phi}}{2}\,\mathbf{i}_y \right)$$

$$= (\bar{A}\,\mathbf{i}_x + j\bar{A}\,\mathbf{i}_y) + (\bar{B}\,\mathbf{i}_x - jB\,\mathbf{i}_y)$$

Thus an elliptically polarized vector can be expressed as the superposition of two circularly polarized vectors of unequal magnitudes and opposite senses of rotation.

4.49. $\langle w_e \rangle = \frac{1}{4}\epsilon_0\,\bar{\mathbf{E}}\cdot\bar{\mathbf{E}}^*$

$$= \frac{1}{4}\epsilon_0\left[\left(-\sqrt{3} - j\tfrac{1}{2}\right)\mathbf{i}_x + \left(1 - j\tfrac{\sqrt{3}}{2}\right)\mathbf{i}_y + j\sqrt{3}\,\mathbf{i}_z\right] e^{-j0.02\pi(\sqrt{3}x + 3y + 2z)}$$

$$\times\left[\left(-\sqrt{3} + j\tfrac{1}{2}\right)\mathbf{i}_x + \left(1 + j\tfrac{\sqrt{3}}{2}\right)\mathbf{i}_y - j\sqrt{3}\,\mathbf{i}_z\right] e^{j0.02\pi(\sqrt{3}x + 3y + 2z)}$$

$$= 2\epsilon_0 .$$

4.50. (a) $\langle w_e \rangle = \frac{1}{4}\epsilon_0\,\bar{\mathbf{E}}\cdot\bar{\mathbf{E}}^* = \frac{1}{4}\epsilon_0\left(10\sin\pi x\, e^{-j\frac{\pi}{2}} e^{-j\sqrt{3}\pi z}\,\mathbf{i}\right)\cdot\left(10\sin\pi x\, e^{j\frac{\pi}{2}} e^{j\sqrt{3}\pi z}\,\mathbf{i}_y\right)$

$$= 25\,\epsilon_0 \sin^2\pi x .$$

(b) $\bar{\mathbf{B}} = \frac{1}{j\omega}\nabla\times\bar{\mathbf{E}} = -\frac{10\sqrt{3}\pi}{6\pi\times 10^8}\sin\pi x\; e^{-j\frac{\pi}{2}} e^{-j\sqrt{3}\pi z}\,\mathbf{i}_x$

$$+ j\,\frac{10\pi}{6\pi\times 10^8}\cos\pi x\; e^{-j\frac{\pi}{2}} e^{-j\sqrt{3}\pi z}\,\mathbf{i}_y$$

$$\langle w_m \rangle = \frac{1}{4\mu_0}\,\bar{\mathbf{B}}\cdot\bar{\mathbf{B}}^* = \frac{10^{-9}}{144\pi}\left(25 + 50\sin^2\pi x\right).$$

(c) $\langle \mathbf{P} \rangle = \mathcal{R}e\left(\frac{1}{2}\bar{\mathbf{E}}\times\frac{\bar{\mathbf{B}}^*}{\mu_0}\right) = \frac{5}{8\sqrt{3}\pi}\sin^2\pi x\,\mathbf{i}_z .$

(d) $\mathcal{I}m\,[\bar{\mathbf{P}}] = \mathcal{I}m\left(\frac{1}{2}\mathbf{E}\times\frac{\bar{\mathbf{B}}^*}{\mu_0}\right) = -\frac{5}{48\pi}\sin 2\pi x\,\mathbf{i}_x .$

CHAPTER 5

5.1. Denoting the equal and opposite velocities to be $\pm v_0 \underset{\sim}{i}_x$ before the application of the field $E_0 \underset{\sim}{i}_x$, we obtain the velocities after the application of the field as $\underset{\sim}{v}_1 = \left(v_0 - \frac{|e| E_0 t}{m}\right) \underset{\sim}{i}_x$ and $\underset{\sim}{v}_2 = -\left(v_0 + \frac{|e| E_0 t}{m}\right) \underset{\sim}{i}_x$. The kinetic energies are

$$\frac{1}{2} m v_1^2 = \frac{1}{2} m v_0^2 - \frac{|e| E_0 t}{2}\left(2v_0 - \frac{|e| E_0 t}{m}\right)$$

$$\frac{1}{2} m v_2^2 = \frac{1}{2} m v_0^2 + \frac{|e| E_0 t}{2}\left(2v_0 + \frac{|e| E_0 t}{m}\right)$$

Thus the gain in kinetic energy by the accelerating electron is greater than the loss in kinetic energy by the decelerating electron.

5.2. (a) $m \frac{d\underset{\sim}{v}_d}{dt} + \frac{m}{\tau} \underset{\sim}{v}_d = e \underset{\sim}{E}_0 \cos \omega t$

$j\omega m \bar{\underset{\sim}{v}}_d + \frac{m}{\tau} \bar{\underset{\sim}{v}}_d = e \underset{\sim}{E}_0$ or, $\bar{\underset{\sim}{v}}_d = \frac{\tau e \underset{\sim}{E}_0}{m\sqrt{1+\omega^2\tau^2}} e^{-j \tan^{-1} \omega\tau}$

which gives the required solution for $\underset{\sim}{v}_d$.

(b) $\mu_e = \frac{\sigma}{N_e |e|} = 0.06506 \text{ m}^2/\text{volt-sec}$, $\tau = \frac{\mu_e m}{|e|} = 3.7 \times 10^{-14}$ sec.

The required frequency is given by $\tan^{-1} \omega\tau = \frac{\pi}{4}$, or, $f = 0.433 \times 10^{13}$ Hz.

The drift velocity magnitude at this frequency is $\frac{\tau |e| E_0}{m\sqrt{2}}$. Hence, the mobility at this frequency is $\frac{\tau |e|}{m\sqrt{2}}$. Since the mobility at zero frequency is $\frac{\tau |e|}{m}$, the required ratio is $\frac{1}{\sqrt{2}}$.

5.3. $\underset{\sim}{E} = -\underset{\sim}{\nabla} V = -\underset{\sim}{\nabla}(50xy) = -50y \underset{\sim}{i}_x - 50x \underset{\sim}{i}_y$.

For $x=0, y>0$, $E_n = [E_x]_{x=0} = -50y$, $\rho_s = -50\epsilon_0 y$

For $y=0, x>0$, $E_n = [E_y]_{y=0} = -50x$, $\rho_s = -50\epsilon_0 x$

For $xy=2$, $\underset{\sim}{i}_n = \pm\left[\frac{\underset{\sim}{\nabla}(xy)}{|\underset{\sim}{\nabla}(xy)|}\right]_{xy=2} = \pm \frac{2\underset{\sim}{i}_x + x^2 \underset{\sim}{i}_y}{\sqrt{x^4+4}}$. We however choose − sign since it gives normal pointing into the region between the conductors.

Then, $E_n = [\underset{\sim}{E}]_{xy=2} \cdot \underset{\sim}{i}_n = \frac{50}{x}\sqrt{x^4+4}$, and $\rho_s = \frac{50\epsilon_0}{4}\sqrt{x^4+4}$.

5.4. $\underset{\sim}{E}_a = \frac{\rho_{L0}}{2\pi\epsilon_0 (y^2+z^2)} (y \underset{\sim}{i}_y + z \underset{\sim}{i}_z)$. For the total field inside the conductor to be zero,

$$[\underset{\sim}{E}_s]_{z<-d} = -[\underset{\sim}{E}_a]_{z<-d} = -\frac{\rho_{L0}}{2\pi\epsilon_0 (y^2+z^2)} (y \underset{\sim}{i}_y + z \underset{\sim}{i}_z).$$

From symmetry considerations, we then have

$$[\underset{\sim}{E}_s]_{z>-d} = -\frac{\rho_{L0}}{2\pi\epsilon_0[y^2+(z+2d)^2]}[y\,\underset{\sim}{i}_y + (z+2d)\,\underset{\sim}{i}_z]$$

The total field in the region $z>-d$ is then given by

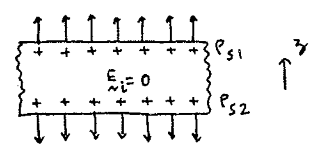
$$[\underset{\sim}{E}]_{z>-d} = \underset{\sim}{E}_a + [\underset{\sim}{E}_s]_{z>-d} = \frac{\rho_{L0}(y\,\underset{\sim}{i}_y + z\,\underset{\sim}{i}_z)}{2\pi\epsilon_0(y^2+z^2)} - \frac{\rho_{L0}[y\,\underset{\sim}{i}_y + (z+2d)\,\underset{\sim}{i}_z]}{2\pi\epsilon_0[y^2+(z+2d)^2]}$$

$$[\underset{\sim}{E}]_{z=-d} = -\frac{\rho_{L0}\,d}{\pi\epsilon_0(y^2+d^2)}\underset{\sim}{i}_z\ ,\quad \rho_s = \epsilon_0[E_z]_{z=-d} = -\frac{\rho_{L0}\,d}{\pi(y^2+d^2)}\ \text{C/m}^2$$

The induced surface charge per unit length along the x direction is $\int_{y=-\infty}^{\infty}\int_{x=0}^{1}\rho_s\,dy\,dx = -\rho_{L0}$. Also, the second term in the expression for $[\underset{\sim}{E}]_{z>-d}$ is the same as the first term with z replaced by $(z+2d)$ and with a negative sign. Hence, it is the field due to a line charge parallel to the actual line charge but passing through $z=-2d$ and having a uniform density same as the negative of the actual line charge. Thus the field outside the conductor is the same as the field due to the actual line charge along the x axis and an image line charge of uniform density $-\rho_{L0}$ C/m situated parallel to the actual line charge and passing through $(0,0,-2d)$.

5.5. Method of solution similar to that of Problem 5.4.

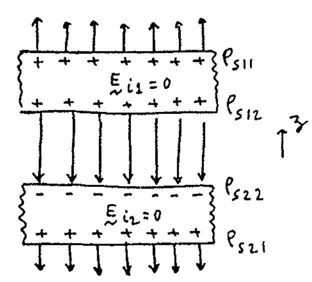
5.6. (a) $\underset{\sim}{E}_i = -\frac{\rho_{s1}}{2\epsilon_0}\underset{\sim}{i}_z + \frac{\rho_{s2}}{2\epsilon_0}\underset{\sim}{i}_z = 0$



$\therefore\ \rho_{s1} = \rho_{s2} = \frac{\rho_{s0}}{2}$.

(b) $\underset{\sim}{E}_{i1} = \left(-\frac{\rho_{s11}}{2\epsilon_0} + \frac{\rho_{s12}}{2\epsilon_0} + \frac{\rho_{s22}}{2\epsilon_0} + \frac{\rho_{s21}}{2\epsilon_0}\right)\underset{\sim}{i}_z = 0$

$\underset{\sim}{E}_{i2} = \left(-\frac{\rho_{s11}}{2\epsilon_0} - \frac{\rho_{s12}}{2\epsilon_0} - \frac{\rho_{s22}}{2\epsilon_0} + \frac{\rho_{s21}}{2\epsilon_0}\right)\underset{\sim}{i}_z = 0$



$\therefore\ -\rho_{s11} + \rho_{s12} + \rho_{s22} + \rho_{s21} = 0$

$-\rho_{s11} - \rho_{s12} - \rho_{s22} + \rho_{s21} = 0$

Solving these equations, we get $\rho_{s21} = \rho_{s11}$, $\rho_{s22} = -\rho_{s12}$.

Then using $\rho_{s11} + \rho_{s12} = \rho_{s1}$, and $\rho_{s21} + \rho_{s22} = \rho_{s2}$, we obtain

$\rho_{s11} = \rho_{s21} = \frac{1}{2}(\rho_{s1} + \rho_{s2})$ and $\rho_{s12} = -\rho_{s22} = \frac{1}{2}(\rho_{s1} - \rho_{s2})$.

5.7. From the symmetry associated with the charge distribution, the electric field must be radially directed. Then choosing Gaussian surfaces which are cylinders having the same axis ($r=0$) as the conductors and of length l, we get $2\pi r l E_r = 0$ for $r<a$ since there is no charge enclosed by the Gaussian surface. Thus $E_r = 0$ for $r<a$. Now, since the field inside the conductor $a<r<b$ is zero, there cannot be any charge on the surface $r=a$. All of the charge associated with the inner conductor resides on the surface $r=b$. Thus $[\rho_s]_{r=a} = 0$, and $[\rho_s]_{r=b} = \frac{\rho_{L1}}{2\pi b}$ c/m^2.

Proceeding further, we have $2\pi r l E_r = \frac{1}{\epsilon_0}\rho_{L1} l$ for $b<r<c$, or $\underset{\sim}{E} = \frac{\rho_{L1}}{2\pi\epsilon_0 r}\underset{\sim}{i}_r$ for $b<r<c$, which together with the fact that the field inside the conductor $c<r<d$ is zero gives

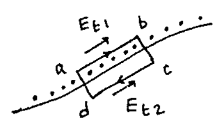
$$[\rho_s]_{r=c} = \epsilon_0[\underset{\sim}{E}]_{r=c}\cdot[-\underset{\sim}{i}_r] = -\frac{\rho_{L1}}{2\pi c} \text{ c/m}^2, \text{ and}$$

$$[\rho_s]_{r=d} = \frac{1}{2\pi d}\left\{\rho_{L2} - [\rho_s]_{r=c}\,2\pi c\right\} = \frac{1}{2\pi d}(\rho_{L2}+\rho_{L1}) \text{ c/m}^2.$$

5.8. Method of solution is similar to that of Problem 5.7.

$$[\rho_s]_{r=a} = 0,\quad [\rho_s]_{r=b} = \frac{Q_1}{4\pi b^2},\quad [\rho_s]_{r=c} = -\frac{Q_1}{4\pi c^2},\quad [\rho_s]_{r=d} = \frac{Q_1+Q_2}{4\pi d^2}$$

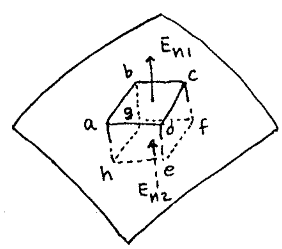
5.9. (a) Considering a rectangular path abcda, we have



$$\lim_{\substack{ad\to 0\\ bc\to 0}} \oint_{abcda} \underset{\sim}{E}\cdot d\underset{\sim}{l} = E_{t1}(ab) + (-E_{t2})(cd)$$

Thus $E_{t1}(ab) - E_{t2}(cd) = 0$ or $E_{t1} = E_{t2}$

(b) Considering a rectangular box abcdefgh, we have, as the side surfaces (ss) tend to zero,



$$\lim_{ss\to 0} \oint \underset{\sim}{E}\cdot d\underset{\sim}{S} = E_{n1}(abcd) - E_{n2}(efgh)$$

$$\lim_{ss\to 0} \int_{\substack{\text{volume}\\ \text{of box}}} \rho\, dv = \rho_s(abcd)$$

Thus $E_{n1}(abcd) - E_{n2}(efgh) = \frac{1}{\epsilon_0}\rho_s(abcd)$, or $E_{n1} - E_{n2} = \frac{1}{\epsilon_0}\rho_s$.

5.10. (a) Considering a rectangular box as in Problem 5.9(b), we have, as the side surfaces (ss) tend to zero,

$$\lim_{ss\to 0} \oint_{\substack{\text{surface}\\ \text{of box}}} \underset{\sim}{B}\cdot d\underset{\sim}{S} = B_{n1}(abcd) - B_{n2}(efgh)$$

Thus $B_{n1}(abcd) - B_{n2}(efgh) = 0$ or $B_{n1} = B_{n2}$

(b) Considering a rectangular path abcda as in Problem 5.9(a), we have

$$\lim_{\substack{ad\to 0\\ bc\to 0}} \oint_{abcda} \underset{\sim}{B}\cdot d\underset{\sim}{l} = B_{t1}(ab) - B_{t2}(cd).$$

$$\lim_{\substack{ad\to 0\\ bc\to 0}} \oint_{\substack{\text{area}\\ abcd}} \underset{\sim}{J}\cdot d\underset{\sim}{S} = J_s(ab), \text{ and } \lim_{\substack{ad\to 0\\ bc\to 0}} \left[\frac{d}{dt}\int_{\substack{\text{area}\\ abcd}} \epsilon_0 \underset{\sim}{E}\cdot d\underset{\sim}{S}\right] = 0$$

Thus $B_{t1}(ab) - B_{t2}(cd) = \mu_0 J_s(ab)$ or $B_{t1} - B_{t2} = \mu_0 J_s$.

5.11. (a) $\underset{\sim}{\nabla}\cdot\underset{\sim}{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(E_\theta \sin\theta) = 0$

$$\underset{\sim}{\nabla}\times\underset{\sim}{E} = \frac{\underset{\sim}{i}_\phi}{r}\left[\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial E_r}{\partial\theta}\right] = 0$$

(b) $[E_\theta]_{r=a} = \left[-E_0\left(1-\frac{a^3}{r^3}\right)\sin\theta\right]_{r=a} = 0$

(c) $[\rho_s]_{r=a} = \epsilon_0[\underset{\sim}{E}]_{r=a}\cdot\underset{\sim}{i}_r = \epsilon_0[E_r]_{r=a} = 3\epsilon_0 E_0\cos\theta$.

(d) $\underset{\sim}{E}_a = [\underset{\sim}{E}]_{a=0} = E_0\cos\theta\,\underset{\sim}{i}_r - E_0\sin\theta\,\underset{\sim}{i}_\theta = E_0\,\underset{\sim}{i}_z$.

$[\underset{\sim}{E}_s]_{r<a} = -\underset{\sim}{E}_a = -E_0\cos\theta\,\underset{\sim}{i}_r + E_0\sin\theta\,\underset{\sim}{i}_\theta$

$[\underset{\sim}{E}_s]_{r>a} = [\underset{\sim}{E}]_{r>a} - \underset{\sim}{E}_a = \frac{E_0 a^3}{r^3}(2\cos\theta\,\underset{\sim}{i}_r + \sin\theta\,\underset{\sim}{i}_\theta)$.

(e) $[\underset{\sim}{E}_s]_{r=a-} = -E_0\cos\theta\,\underset{\sim}{i}_r + E_0\sin\theta\,\underset{\sim}{i}_\theta$

$[\underset{\sim}{E}_s]_{r=a+} = 2E_0\cos\theta\,\underset{\sim}{i}_r + E_0\sin\theta\,\underset{\sim}{i}_\theta$

Thus the θ components are equal and the r components are discontinuous by $3E_0\cos\theta$ which is equal to $\frac{1}{\epsilon_0}\rho_s$.

5.12. Charge in the electron cloud of He is 4e.

$$d = \frac{4\pi\epsilon_0 a^3}{Q}E_0 = 4\pi\times\frac{10^{-9}}{36\pi}\times\frac{10^{-30}}{6.4\times10^{-19}}\times5\times10^6 = 0.87\times10^{-15}\text{ m}$$

$$\frac{d}{a} = \frac{0.87\times10^{-15}}{10^{-10}} = 0.87\times10^{-5}.$$

5.13. Let the displacement be d. With reference to the notation of Figure 5.9 of the text, the electric field at the nucleus due to the electron cloud is given by

$$\underset{\sim}{E}_z = \frac{1}{\epsilon_0}\,\frac{4\pi\int_{r=0}^{d}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\rho(r)\,r^2\sin\theta\,dr\,d\theta\,d\phi}{4\pi d^2}\,\underset{\sim}{i}_z$$

For small d,

$$\underset{\sim}{E}_z \approx \frac{1}{\epsilon_0}\,\frac{4\pi\rho(0)\int_{r=0}^{d} r^2\,dr}{4\pi d^2}\,\underset{\sim}{i}_z = \frac{\rho(0)\,d}{3\epsilon_0}\,\underset{\sim}{i}_z.$$

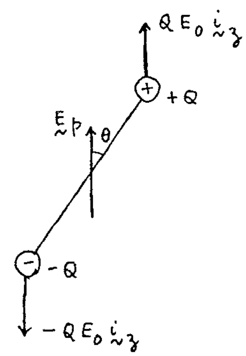
For equilibrium displacement, we have

$$Q E_0 \mathbf{i}_z + \frac{Q\,\rho(0)\,d}{3\epsilon_0}\mathbf{i}_z = 0 \qquad \text{or} \qquad d = -\frac{3\epsilon_0 E_0}{\rho(0)} = \frac{3\epsilon_0 E_0}{|\rho(0)|}$$

$$\mathbf{p}_e = Q d\,\mathbf{i}_z = \frac{3\epsilon_0 Q E_0}{|\rho(0)|}\mathbf{i}_z \quad \text{and} \quad \alpha_e = \frac{3\epsilon_0 Q}{|\rho(0)|}.$$

For uniformly charged cloud, $|\rho(0)| = \dfrac{Q}{\frac{4}{3}\pi a^3}$, $\alpha_e = 4\pi\epsilon_0 a^3$.

5.14. The electric field $\mathbf{E}_p = E_0\mathbf{i}_z$ exerts equal and opposite forces on the positive and negative charges constituting the dipole. Thus



$T = (Q E_0 \sin\theta)\,d = p E_0 \sin\theta$

or $\mathbf{T} = \mathbf{p} \times \mathbf{E}_p$. The torque is zero for two orientations of the dipole, one along the field and the other opposite to the field. However, if the dipole is turned about its center in either direction, the resulting torque acts to tu n it further away in the latter case whereas it tends to bring it back to equilibrium in the former case. Thus the torque tends to align the dipole with the field.

5.15. Method similar to that of Example 5-4.

5.16. Denoting the charge density to be $\rho(\mathbf{r}')$, we write the electric field intensity at a point $P(\mathbf{r})$ due to the volume charge distribution in the spherical volume V' of radius a as

$$\mathbf{E}(\mathbf{r}) = \int_{V'} \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}')\,dv'$$

Then $$\mathbf{E}_{av} = \frac{1}{\frac{4}{3}\pi a^3}\int_V \mathbf{E}(\mathbf{r})\,dv = \frac{1}{\frac{4}{3}\pi a^3}\int_V\int_{V'} \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}')\,dv'\,dv$$

$$= -\frac{1}{\frac{4}{3}\pi a^3}\int_{V}\rho(\mathbf{r}')\left[\int_V \frac{\mathbf{r}'-\mathbf{r}}{4\pi\epsilon_0|\mathbf{r}'-\mathbf{r}|^3}\,dv\right]dv'$$

But the quantity inside the brackets is the electric field intensity at $Q(\mathbf{r}')$ due to a charge distribution in the spherical volume with uniform density 1 c/m³. From Gauss' law, this can be obtained as $\frac{1}{3\epsilon_0}\mathbf{r}'$. Thus

$$\mathbf{E}_{av} = -\frac{1}{4\pi a^3\epsilon_0}\int_{V'}\rho(\mathbf{r}')\,\mathbf{r}'\,dv' = -\frac{\mathbf{p}}{4\pi\epsilon_0 a^3}.$$

5.17. (a) $\underset{\sim}{E}_a = \dfrac{Q}{4\pi\epsilon_0 r^2}\,\underset{\sim}{i}_r$. Hence, the density of the induced dipole moments is inversely proportional to r^2. Since the surface area of a sphere is proportional to r^2, the positive charge associated with the dipoles in one spherical shell of infinitesimal thickness is the same as the negative charge associated with the dipoles in the adjacent spherical shell. Hence, the polarization volume charge density is zero.

Let $[\rho_{ps}]_{r=b} = \rho_{pso}$. Then $[\rho_{ps}]_{r=a} = -\rho_{pso}\dfrac{b^2}{a^2}$.

$\underset{\sim}{E}_s = -\dfrac{\rho_{pso}}{\epsilon_0}\dfrac{b^2}{r^2}\,\underset{\sim}{i}_r$ for $a<r<b$, 0 otherwise

$\underset{\sim}{E} = \underset{\sim}{E}_a + \underset{\sim}{E}_s = \left(\dfrac{Q}{4\pi\epsilon_0 r^2} - \dfrac{\rho_{pso}b^2}{\epsilon_0 r^2}\right)\underset{\sim}{i}_r$ for $a<r<b$, $\underset{\sim}{E}_a$ otherwise

$\underset{\sim}{P} = \epsilon_0\chi_{eo}\underset{\sim}{E} = \chi_{eo}\left(\dfrac{Q}{4\pi r^2} - \dfrac{\rho_{pso}b^2}{r^2}\right)\underset{\sim}{i}_r$

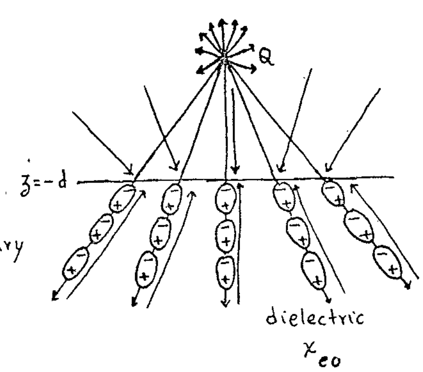
$\rho_{ps} = \underset{\sim}{P}\cdot\underset{\sim}{i}_n = \chi_{eo}\left(\dfrac{Q}{4\pi b^2} - \rho_{pso}\right)$ for $r=b$ and $-\chi_{eo}\left(\dfrac{Q}{4\pi a^2} - \rho_{pso}\dfrac{b^2}{a^2}\right)$ for $r=a$.

Thus $\rho_{pso} = \chi_{eo}\left(\dfrac{Q}{4\pi b^2} - \rho_{pso}\right)$. Solving this equation for ρ_{pso}, we obtain the given expressions for the polarization surface charge densities.

(b) $\underset{\sim}{E}$ is same as $\underset{\sim}{E}_a$ for $r<a$ and $r>b$. For $a<r<b$, substituting for ρ_{pso}, we obtain $\underset{\sim}{E} = \dfrac{Q}{4\pi\epsilon_0(1+\chi_{eo})r^2}\,\underset{\sim}{i}_r$.

(c) For $a\to 0$ and $b\to\infty$, $\underset{\sim}{E} = \dfrac{Q}{4\pi\epsilon_0(1+\chi_{eo})r^2}\,\underset{\sim}{i}_r$ for $0<r<\infty$. Since $\chi_{eo} > 1$, this field is smaller than the field due to the point charge if the medium were free space. A portion of the point charge is neutralized by the polarization charges.

5.18. $\underset{\sim}{E}_a = \dfrac{Q}{4\pi\epsilon_0 r^2}\,\underset{\sim}{i}_r$. This applied field induces dipole moments in the dielectric as shown in the figure, resulting in a polarization surface charge on the surface $z=-d$. As long as this polarization surface charge produces a secondary field in the dielectric which has the same spatial dependence as the applied field, the polarization volume charge density inside the



dielectric is zero. We will assume this to be true and later show that the assumption is consistent with the results. Thus let

$$\underset{\sim}{E}_s = -\frac{kQ}{4\pi\epsilon_0 r^2}\underset{\sim}{i}_r = -\frac{kQ}{4\pi\epsilon_0(r_c^2+z^2)^{3/2}}(r_c\underset{\sim}{i}_r + z\underset{\sim}{i}_z) \quad \text{for } z<-d$$

Then from symmetry con iderations,

$$\underset{\sim}{E}_s = -\frac{kQ}{4\pi\epsilon_0[r_c^2+(z+2d)^2]^{3/2}}[r_c\underset{\sim}{i}_r + (z+2d)\underset{\sim}{i}_z] \quad \text{for } z>-d$$

Writing the expressions for the total fields on both sides of $z=-d$ and evaluating ρ_{ps} from boundary condition, we obtain $\rho_{ps} = -\dfrac{kQd}{2\pi(r_c^2+d^2)^{3/2}}$.

But $\rho_{ps} = \underset{\sim}{P}\cdot\underset{\sim}{i}_n = [\epsilon_0\chi_e\underset{\sim}{E}]_{z=-d_-}\cdot\underset{\sim}{i}_z = -\dfrac{\chi_{e0}Q(1-k)d}{4\pi(r_c^2+d^2)^{3/2}}$. Equating the two expressions for ρ_{ps} and evaluating k, we get $k = \dfrac{\chi_{e0}}{2+\chi_{e0}}$ which gives the required expression for ρ_{ps}. Since k is a constant, the secondary field has the same spatial dependence as the applied field inside the dielectric. Hence, the polarization volume charge density is indeed zero. Proceeding further, we get

$$\underset{\sim}{E} = \begin{cases} \dfrac{2Q/(2+\chi_{e0})}{4\pi\epsilon_0 r^2}\underset{\sim}{i}_r & \text{for } z<-d \\[2ex] \dfrac{Q(r_c\underset{\sim}{i}_r + z\underset{\sim}{i}_z)}{4\pi\epsilon_0(r_c^2+z^2)^{3/2}} - \dfrac{[\chi_{e0}Q/(2+\chi_{e0})][r_c\underset{\sim}{i}_r+(z+2d)\underset{\sim}{i}_z]}{4\pi\epsilon_0[r_c^2+(z+2d)^2]^{3/2}} & \text{for } z>-d \end{cases}$$

as required to be shown by the problem.

5.19. (a) $\underset{\sim}{\nabla}\cdot\underset{\sim}{E} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r E_r) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}(\sin\theta\, E_\theta) = 0$ for both $\underset{\sim}{E}_o$ and $\underset{\sim}{E}_i$

$\underset{\sim}{\nabla}\times\underset{\sim}{E} = \dfrac{\underset{\sim}{i}_\phi}{r}\left[\dfrac{\partial}{\partial r}(r E_\theta) - \dfrac{\partial E_r}{\partial\theta}\right] = 0$ for both $\underset{\sim}{E}_o$ and $\underset{\sim}{E}_i$.

(b) See page 535 of the text for answers.

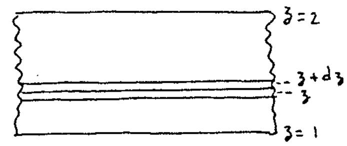
(c) $[E_{\theta s}]_{r=a+} = [E_{\theta s}]_{r=a-} = \dfrac{\chi_{e0}}{3+\chi_{e0}}E_0\sin\theta$.

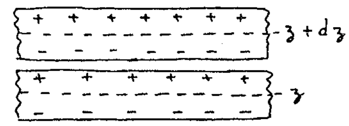
(d) $[\rho_{ps}]_{r=a} = \epsilon_0\left\{[E_{rs}]_{r=a+} - [E_{rs}]_{r=a-}\right\} = \dfrac{3\chi_{e0}}{3+\chi_{e0}}\epsilon_0 E_0\cos\theta$

(e) $\underset{\sim}{P} = \epsilon_0\chi_{e0}\underset{\sim}{E}_i = \dfrac{3\epsilon_0\chi_{e0}}{3+\chi_{e0}}(E_0\cos\theta\,\underset{\sim}{i}_r - E_0\sin\theta\,\underset{\sim}{i}_\theta)$

$[\rho_{ps}]_{r=a} = [\underset{\sim}{P}]_{r=a}\cdot\underset{\sim}{i}_r = \dfrac{3\chi_{e0}}{3+\chi_{e0}}\epsilon_0 E_0\cos\theta$.

5.20. We divide the slab into a series of slabs of infinitesimal thickness dz so that the susceptibility within each slab can be considered to be constant. If we now consider two such slabs centered at z and $z + dz$ and apply the results of Example 5-5 to each of these slabs, we get polarization surface charge densities





$$\pm \frac{\epsilon_0 \chi_e(z)}{1+\chi_e(z)} E_0 \text{ and } \pm \frac{\epsilon_0 \chi_e(z+dz)}{1+\chi_e(z+dz)} \text{ or } \pm \epsilon_0 \frac{z}{4} E_0 \text{ and } \pm \epsilon_0 \frac{z+\Delta z}{4} E_0,$$

respectively. These are however not polarization surface charge densities except at the surfaces $z=1$ and $z=2$. For $1<z<2$, the unequal positive and negative polarization charge associated with adjacent slabs are equivalent to a polarization volume charge. The polarization volume charge density is $\left(\epsilon_0 \frac{z}{4} E_0 - \epsilon_0 \frac{z+dz}{4} E_0\right) \frac{1}{dz}$ or $-\epsilon_0 \frac{E_0}{4}$. Thus $\rho_p = -\frac{\epsilon_0 E_0}{4}$ for $1<z<2$, and $\rho_{ps} = -\frac{\epsilon_0 E_0}{4}$ for $z=1$ and $\frac{\epsilon_0 E_0}{2}$ for $z=2$.

The total polarization volume charge is equal to zero.

The secondary field is the field produced by the polarization charge distribution. This is given by $\underset{\sim}{E}_s = -\frac{E_0 z}{4} \underset{\sim}{i}_z$ for $1<z<2$ and 0 otherwise.

The total electric field is then given by

$\underset{\sim}{E} = E_0\left(1-\frac{z}{4}\right)\underset{\sim}{i}_z$ for $1<z<2$ and $E_0 \underset{\sim}{i}_z$ otherwise.

For $\underset{\sim}{E}_a = E_0 \cos \omega t \, \underset{\sim}{i}_z$, following the method of Example 5-6 and taking into account the polarization volume charge in addition to the polarization surface charge, we get $\underset{\sim}{P} = \frac{\epsilon_0 z E_0}{4} \cos \omega t \, \underset{\sim}{i}_z$.

5.21. Let the two conducting plates be in the planes $z=$ and $z=$. For each case, we find $\underset{\sim}{E}$ and then use $V = \int_{z=d}^{0} \underset{\sim}{E} \cdot d\underset{\sim}{l}$

(a) $\underset{\sim}{E} = -\frac{\rho_{s0}}{\epsilon_0} \underset{\sim}{i}_z$ for $0<z<d$, $V = \frac{\rho_{s0} d}{\epsilon_0}$

(b) $\underset{\sim}{E} = -\frac{\rho_{s0}}{4\epsilon_0} \underset{\sim}{i}_z$ for $0<z<d$, $V = \frac{\rho_{s0} d}{4\epsilon_0}$

(c) $\underset{\sim}{E} = -\frac{\rho_{s0}}{2\epsilon_0} \underset{\sim}{i}_z$ for $0<z<t$ and $-\frac{\rho_{s0}}{4\epsilon_0} \underset{\sim}{i}_z$ for $t<z<d$, $V = \frac{\rho_{s0}}{4\epsilon_0}(d+t)$

(d) $\underset{\sim}{E} = -\frac{\rho_{s0}}{\epsilon_1 + (\epsilon_2-\epsilon_1) z/d} \underset{\sim}{i}_z$ for $0<z<d$, $V = \frac{\rho_{s0} d}{\epsilon_2 - \epsilon_1} \ln \frac{\epsilon_2}{\epsilon_1}$.

5.22. For each case, we assume the surface charge densities to be $\pm \rho_{s0}$ and find V as in Problem 5.21, and equate it to V_0 to obtain ρ_{s0}

(a) $\dfrac{\epsilon_0 V_0}{d}$ (b) $\dfrac{4\epsilon_0 V_0}{d}$ (c) $\dfrac{4\epsilon_0 V_0}{d+t}$ (d) $\dfrac{(\epsilon_2-\epsilon_1)V_0}{d \ln \epsilon_2/\epsilon_1}$.

5.23. (a) Since the dielectric slab is a plane slab and since the permittivity is a function of z only, there is no secondary field outside the dielectric. Hence, $\mathbf{D}_0 = \epsilon_0 \mathbf{E}_a = \epsilon_0 E_0 \mathbf{i}_z$.

(b) Since there are no true charges in the dielectric, $\mathbf{D}_i = \mathbf{D}_0 = \epsilon_0 E_0 \mathbf{i}_z$.

(c) $\mathbf{E}_i = \mathbf{D}_i/\epsilon$

(d) $\mathbf{P} = \mathbf{D}_i - \epsilon_0 \mathbf{E}_i$

(e) $\rho_{ps} = \mathbf{P}\cdot\mathbf{i}_n = [\mathbf{P}]_{z=0}\cdot(-\mathbf{i}_z)$ for $z=0$, $[\mathbf{P}]_{z=d}\cdot\mathbf{i}_z$ for $z=d$.

(f) $\rho_p = -\nabla\cdot\mathbf{P}$

For answers to (c), (d), (e), and (f), see page 536 of the text.

5.24. Since $\mathbf{E}$ for $a<r<b$ is independent of θ and ϕ, the permittivity of the dielectric must be a function of r only. Hence

(a) $\mathbf{D} = \dfrac{Q}{4\pi r^2}\mathbf{i}_r$ for all r, $\mathbf{E} = \dfrac{Q}{4\pi\epsilon r^2}\mathbf{i}_r$ for $a<r<b$ which upon comparison with the given field yields $\epsilon = \epsilon_0 \dfrac{b^2}{r^2}$.

(b) $\mathbf{P} = \mathbf{D} - \epsilon_0\mathbf{E} = \dfrac{Q}{4\pi}\left(\dfrac{1}{r^2}-\dfrac{1}{b^2}\right)\mathbf{i}_r$.

$\rho_{ps} = \mathbf{P}\cdot\mathbf{i}_n = \dfrac{Q}{4\pi}\left(\dfrac{1}{b^2}-\dfrac{1}{a^2}\right)$ for $r=a$, 0 for $r=b$

(c) $\rho_p = -\nabla\cdot\mathbf{P} = \dfrac{Q}{2\pi r b^2}$.

5.25. Applying Faraday's law to the orbital path, we have

$2\pi r\, E_\phi = -\dfrac{d}{dt}(\pi a^2 B_z)$ or $E_\phi = -\dfrac{a}{2}\cdot\dfrac{dB_z}{dt}$.

The force exerted on the electron is then given by

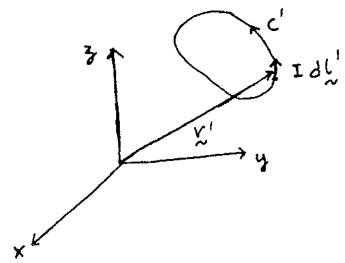
$\mathbf{F} = -|e|E_\phi\,\mathbf{i}_\phi$ or $ma\dfrac{d\omega}{dt}\mathbf{i}_\phi = \dfrac{|e|a}{2}\dfrac{dB_z}{dt}\mathbf{i}_\phi$. Hence, $d\omega = \dfrac{|e|}{2m}dB_z$.

For the application of a magnetic field $\mathbf{B}_m = B_0\mathbf{i}_z$, $dB = B_0$ and $d\omega = \dfrac{|e|}{2m}B_0$.

Thus $\omega-\omega_0 = \dfrac{|e|}{2m}B_0$ for orbit in the positive ϕ direction, and

$-\omega-(-\omega_0) = \dfrac{|e|}{2m}B_0$ or $\omega-\omega_0 = -\dfrac{|e|}{2m}B_0$ for orbit in the negative ϕ direction. Combining the two cases, we get $\omega-\omega_0 = \pm\dfrac{|e|B_0}{2m}$ which is the same as the result given by Eq. (5-98) of the text.

5.26. The torque about the origin acting on the loop is given by



$$\mathbf{T}_0 = I \oint_{C'} \mathbf{r}' \times (d\mathbf{l}' \times \mathbf{B}_m)$$

$$= I \oint_{C'} (\mathbf{r}' \cdot \mathbf{B}_m)\, d\mathbf{l}' - I\, \mathbf{B}_m \oint_{C'} \mathbf{r}' \cdot d\mathbf{l}'$$

But $\oint_{C'} \mathbf{r}' \cdot d\mathbf{l}' = \oint_{C'} d\left(\frac{x'^2}{2} + \frac{y'^2}{2} + \frac{z'^2}{2}\right) = 0$.

Also, from Eq. (3-93) of the text,

$$(\mathbf{r}' \cdot \mathbf{B}_m)\, d\mathbf{l}' = \frac{1}{2} \mathbf{B}_m \times (d\mathbf{l}' \times \mathbf{r}') + \frac{1}{2} d\left[(\mathbf{B}_m \cdot \mathbf{r}')\, \mathbf{r}'\right]$$

Thus
$$\mathbf{T}_0 = I \oint_{C'} \frac{1}{2} \mathbf{B}_m \times (d\mathbf{l}' \times \mathbf{r}') + \frac{1}{2} I \oint_{C'} d\left[(\mathbf{B}_m \cdot \mathbf{r}')\, \mathbf{r}'\right]$$

$$= \mathbf{B}_m \times \left[\frac{1}{2} \oint_{C'} (I\, d\mathbf{l}' \times \mathbf{r}')\right] = \left[\frac{1}{2} \oint_{C'} (\mathbf{r}' \times I\, d\mathbf{l}')\right] \times \mathbf{B}_m = \mathbf{m} \times \mathbf{B}_m$$

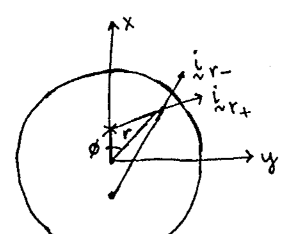
The torque about an arbitrary point defined by the position vector $\mathbf{r}$ is

$$\mathbf{T} = I \oint_{C'} (\mathbf{r}' - \mathbf{r}) \times (d\mathbf{l}' \times \mathbf{B}_m)$$

$$= I \oint_{C'} \mathbf{r}' \times (d\mathbf{l}' \times \mathbf{B}_m) - I \oint_{C'} \mathbf{r} \times (d\mathbf{l}' \times \mathbf{B}_m)$$

$$= I \oint_{C'} \mathbf{r}' \times (d\mathbf{l}' \times \mathbf{B}_m) - I\, \mathbf{r} \times \left(\oint_{C'} d\mathbf{l}'\right) \times \mathbf{B}_m = I \oint_{C'} \mathbf{r}' \times (d\mathbf{l}' \times \mathbf{B}_m) = \mathbf{m} \times \mathbf{B}_m .$$

5.27. Considering the positive z directed current first, we have



$$\mathbf{B}_+ = \frac{\mu_0 I}{2\pi} \frac{1}{\left(r^2 + \frac{d^2}{4} - rd\cos\phi\right)^{1/2}}\, \mathbf{i}_z \times \mathbf{i}_{r_+}$$

The volume integral of $\mathbf{B}_+$ per unit length along the z direction is

$$\int_V \mathbf{B}_+\, dv = -\frac{\mu_0 I}{2\pi} \mathbf{i}_z \times \left[-\int_S \frac{1}{\left(r^2 + \frac{d^2}{4} - rd\cos\phi\right)^{1/2}}\, \mathbf{i}_{r_+}\, dS\right]$$

where S is the crosssectional area of the cylindrical volume. The quantity inside the brackets is the electric field intensity produced at $(\frac{d}{2}, 0, 0)$ by a volume charge of density $2\pi\epsilon_0$ C/m3 in the cylindrical volume. From Gauss' law, this can be obtained as $\frac{\pi d}{2} \mathbf{i}_x$ so that $\int_V \mathbf{B}_+ dv = -\mu_0 \frac{Id}{4} \mathbf{i}_y$.

Similarly, considering the negative z directed current, we get

$\int_V \mathbf{B}_- \, dv = -\frac{\mu_0 I d}{4} \mathbf{i}_y$. Finally,

$$\mathbf{B}_{av} = \frac{1}{\pi a^2} \int_V (\mathbf{B}_+ + \mathbf{B}_-)\, dv = -\frac{\mu_0 I d}{2\pi a^2} \mathbf{i}_y .$$

5.28. Denoting the current density to be $\mathbf{J}(\mathbf{r}')$, the magnetic flux density at a point $P(\mathbf{r})$ due to the volume current distribution in the spherical volume v' of radius a is given by

$$\mathbf{B}(\mathbf{r}) = \int_{v'} \frac{\mu_0 \mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{4\pi |\mathbf{r} - \mathbf{r}'|} \, dv'$$

Then $\mathbf{B}_{av} = \dfrac{1}{\frac{4}{3}\pi a^3} \displaystyle\int_V \mathbf{B}(\mathbf{r})\, dv = \dfrac{1}{\frac{4}{3}\pi a^3} \displaystyle\int_{v'} \dfrac{\mu_0}{4\pi} \mathbf{J}(\mathbf{r}') \times \left[- \displaystyle\int_V \dfrac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|^3}\, dv \right] dv'$

The quantity inside the brackets is the electric field intensity at $Q(\mathbf{r}')$ due to a charge distribution in the spherical volume with uniform density $4\pi\epsilon_0$ C/m³. Evaluating this by using Gauss' law and substituting, we get

$$\mathbf{B}_{av} = \frac{\mu_0}{2\pi a^3} \left[\frac{1}{2} \int_{v'} \mathbf{r}' \times \mathbf{J}(\mathbf{r}')\, dv' \right] = \frac{\mu_0 \mathbf{m}}{2\pi a^3}.$$

5.29. (a) $\mathbf{B}_a = \dfrac{\mu_0 I}{2\pi r} \mathbf{i}_\phi$. Hence, the density of the induced dipole moments is inversely proportional to r. Since the circumference of a circle is proportional to r, the negative z directed current associated with the dipoles in one cylindrical shell of infinitesimal thickness is the same as the positive z directed current associated with the dipoles in the adjacent cylindrical shell. Hence, the magnetization volume current density is zero. Let $[\mathbf{J}_{ms}]_{r=a} = J_{mso} \mathbf{i}_z$. Then $[\mathbf{J}_{ms}]_{r=b} = -J_{mso} \frac{a}{b} \mathbf{i}_z$.

$\mathbf{B}_s = \mu_0 J_{mso} \dfrac{a}{r} \mathbf{i}_\phi$ for $a < r < b$, 0 otherwise.

$\mathbf{B} = \mathbf{B}_a + \mathbf{B}_s = \left(\dfrac{\mu_0 I}{2\pi r} + \mu_0 J_{mso} \dfrac{a}{r} \right) \mathbf{i}_\phi$ for $a < r < b$, $\mathbf{B}_a$ otherwise

$$\mathbf{M} = \frac{\chi_m}{1+\chi_m} \frac{\mathbf{B}}{\mu_0} = \frac{\chi_{mo}}{1+\chi_{mo}} \left(\frac{I}{2\pi r} + J_{mso} \frac{a}{r} \right) \mathbf{i}_\phi$$

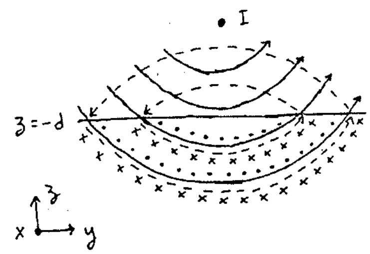
$$\mathbf{J}_{ms} = \mathbf{M} \times \mathbf{i}_n = \begin{cases} \dfrac{\chi_{mo}}{1+\chi_{mo}} \left(\dfrac{I}{2\pi a} + J_{mso} \right) \mathbf{i}_z & \text{for } r = a \\ -\dfrac{\chi_{mo}}{1+\chi_{mo}} \left(\dfrac{I}{2\pi b} + J_{mso} \dfrac{a}{b} \right) \mathbf{i}_z & \text{for } r = b \end{cases}$$

Thus $J_{mso} = \dfrac{\chi_{mo}}{1+\chi_{mo}} \left(\dfrac{I}{2\pi a} + J_{mso} \right)$. Solving this equation for J_{mso}, we obtain the given expressions for the magnetization surface current densities.

(b) $\mathbf{B}$ same as $\mathbf{B}_a$ for $r < a$ and $r > b$. For $a < r < b$, substituting for J_{mso}, we obtain $\mathbf{B} = \mu_0 (1 + \chi_{mo}) \dfrac{I}{2\pi r} \mathbf{i}_\phi$.

(c) For $a \to 0$ and $b \to \infty$, $\mathbf{B} = \mu_0 (1 + \chi_{mo}) \frac{I}{2\pi r} \mathbf{i}_\phi$ for $0 < r < \infty$. The line current is aided or partially neutralized by the magnetization current depending upon whether χ_{mo} is positive or negative.

5.30. $\mathbf{B}_a = \dfrac{\mu_0 I}{2\pi (y^2 + z^2)} (-z\,\mathbf{i}_y + y\,\mathbf{i}_z)$. This applied field induces dipole moments in the magnetic material as shown in the figure, resulting in a magnetization surface current on the surface $z = -d$. As long as this magnetization surface current produces a secondary field in the material which has the same spatial dependence as the applied field, the magnetization volume current density inside the material is zero. We will assume this to be true and later show that the assumption is consistent with the results. Thus let



$$\mathbf{B}_s = k \frac{\mu_0 I}{2\pi (y^2 + z^2)} (-z\,\mathbf{i}_y + y\,\mathbf{i}_z) \quad \text{for } z < -d$$

Then from symmetry considerations

$$\mathbf{B}_s = k \frac{\mu_0 I}{2\pi [y^2 + (z + 2d)^2]} [-(z + 2d)\,\mathbf{i}_y + y\,\mathbf{i}_z] \quad \text{for } z > -d$$

Writing the expressions for the total fields on both sides of $z = -d$ and evaluating $\mathbf{J}_{ms}$ from boundary condition, we obtain $\mathbf{J}_{ms} = \dfrac{kId}{\pi (y^2 + d^2)} \mathbf{i}_x$.

But $\mathbf{J}_{ms} = \mathbf{M} \times \mathbf{i}_n = \left[\dfrac{\chi_{mo}}{1 + \chi_{mo}} \dfrac{\mathbf{B}}{\mu_0} \right]_{z = -d_-} \times \mathbf{i}_z = \dfrac{\chi_{mo}}{1 + \chi_{mo}} \dfrac{I(1 + k)d}{2\pi (y^2 + d^2)} \mathbf{i}_x$.

Equating the two expressions for $\mathbf{J}_{ms}$ and evaluating k, we get $k = \dfrac{\chi_{mo}}{2 + \chi_{mo}}$, which gives the required expression for $\mathbf{J}_{ms}$. Since k is a constant, the secondary field has the same spatial dependence as the applied field inside the material. Hence the magnetization volume current density is indeed zero. Proceeding further, we get

$$\mathbf{B} = \begin{cases} \dfrac{\mu_0 I (2 + 2\chi_{mo})/(2 + \chi_{mo})}{2\pi (y^2 + z^2)} (-z\,\mathbf{i}_y + y\,\mathbf{i}_z) & \text{for } z < -d \\[2ex] \dfrac{\mu_0 I (-z\,\mathbf{i}_y + y\,\mathbf{i}_z)}{2\pi (y^2 + z^2)} + \dfrac{[\chi_{mo} \mu_0 I/(2 + \chi_{mo})][-(z + 2d)\mathbf{i}_y + y\,\mathbf{i}_z]}{2\pi [y^2 + (z + 2d)^2]} & \text{for } z > -d \end{cases}$$

as required to be shown by the problem.

5.31. (a) $\nabla \cdot \underset{\sim}{B} = \frac{1}{r^2}\frac{\partial}{\partial r}(r B_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, B_\theta) = 0$ for both $\underset{\sim}{B}_0$ and $\underset{\sim}{B}_i$.

$\nabla \times \underset{\sim}{B} = \frac{\underset{\sim}{i}_\phi}{r}\left[\frac{\partial}{\partial r}(r B_\theta) - \frac{\partial E_r}{\partial \theta}\right] = 0$ for both $\underset{\sim}{B}_0$ and $\underset{\sim}{B}_i$.

(b) see page 536 of the text for answers.

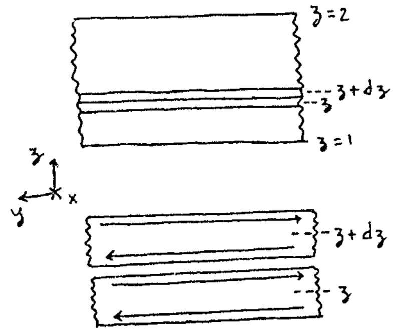
(c) $[B_{rs}]_{r=a+} = [B_{rs}]_{r=a-} = \frac{2\chi_{mo}}{3+\chi_{mo}} B_0 \cos\theta$

(d) $[J_{s\phi}]_{r=a} = \frac{1}{\mu_0}\left\{[B_{\theta s}]_{r=a+} - [B_{\theta s}]_{r=a-}\right\} = \frac{3\chi_{mo}}{3+\chi_{mo}}\frac{B_0}{\mu_0}\sin\theta$

(e) $\underset{\sim}{M} = \frac{\chi_{mo}}{1+\chi_{mo}}\frac{\underset{\sim}{B}_i}{\mu_0} = \frac{3\chi_{mo}}{3+\chi_{mo}}\frac{B_0}{\mu_0}(\cos\theta\, \underset{\sim}{i}_r - \sin\theta\, \underset{\sim}{i}_\theta)$

$[\underset{\sim}{J}_s]_{r=a} = [\underset{\sim}{M}]_{r=a} \times \underset{\sim}{i}_r = \frac{3\chi_{mo}}{3+\chi_{mo}}\frac{B_0}{\mu_0}\sin\theta.$

5.32. We divide the slab into a series of slabs of infinitesimal thickness dz so that the susceptibility within each slab can be considered to be constant. If we now consider two such slabs centered at z and $z+dz$ and apply the results of Example 5-11 to each of these slabs, we get magnetization surface current densities $\pm \frac{\chi_m(z)}{\mu_0} B_0 \underset{\sim}{i}_y$ and $\pm \frac{\chi_m(z+dz)}{\mu_0} B_0 \underset{\sim}{i}_y$ or $\pm \frac{z}{4\mu_0} B_0 \underset{\sim}{i}_y$ and $\pm \frac{z+dz}{4\mu_0} B_0 \underset{\sim}{i}_y$ respectively. These are however not magnetization surface current densities except at the surfaces $z=1$ and $z=2$. For $1<z<2$, the unequal positive y directed and negative y directed magnetization currents associated with adjacent slabs are equivalent to magnetization volume current. The magnetization volume current density is $\left(-\frac{z}{4\mu_0}B_0\underset{\sim}{i}_y + \frac{z+dz}{4\mu_0}B_0\underset{\sim}{i}_y\right)\frac{1}{dz}$ or $\frac{B_0}{4\mu_0}\underset{\sim}{i}_y$. Thus



$\underset{\sim}{J}_m = \frac{B_0}{4\mu_0}\underset{\sim}{i}_y$ for $1<z<2$ and $\underset{\sim}{J}_{ms} = \frac{B_0}{4\mu_0}\underset{\sim}{i}_y$ for $z=1$ and $-\frac{B_0}{2\mu_0}\underset{\sim}{i}_y$ for $z=2$.

The total magnetization current crossing a $y=$ constant plane is equal to zero. The secondary field is the field produced by the magnetization current distribution. This is given by $\underset{\sim}{B}_s = \frac{B_0 z}{4}\underset{\sim}{i}_x$ for $1<z<2$ and 0 otherwise. The total magnetic field is then given by

$\underset{\sim}{B} = B_0\left(1+\frac{z}{4}\right)\underset{\sim}{i}_x$ for $1<z<2$ and $B_0 \underset{\sim}{i}_z$ otherwise.

5.33. Let the conducting plates be in the $z=0$ and $z=d$ planes and let the current flow be in the x direction. For each case, we find $\underset{\sim}{B}$ and then use

$$\psi = \int \underset{\sim}{B} \cdot d\underset{\sim}{S} = \int_{z=0}^{d} B_y \, dz .$$

(a) $\underset{\sim}{B} = \mu_0 J_{S0} \underset{\sim}{i}_y$ for $0<z<d$, $\psi = \mu_0 J_{S0} d$

(b) $\underset{\sim}{B} = 4\mu_0 J_{S0} \underset{\sim}{i}_y$ for $0<z<d$, $\psi = 4\mu_0 J_{S0} d$

(c) $\underset{\sim}{B} = 2\mu_0 J_{S0} \underset{\sim}{i}_y$ for $0<z<t$ and $4\mu_0 J_{S0} \underset{\sim}{i}_y$ for $t<z<d$, $\psi = \mu_0 J_{S0}(4d-2t)$

(d) $\underset{\sim}{B} = \left[\mu_1 + (\mu_2-\mu_1)\frac{z}{d}\right] J_{S0} \underset{\sim}{i}_y$, $\psi = \dfrac{\mu_1+\mu_2}{2} J_{S0} d$.

5.34. For each case, we assume the surface current densities to be $\pm J_{S0} \underset{\sim}{i}_x$ and find ψ as in Problem 5.33, and equate it to ψ_0 to obtain J_{S0}.

(a) $\psi_0/\mu_0 d$ (b) $\psi_0/4\mu_0 d$ (c) $\psi_0/\mu_0(4d-2t)$ (d) $2\psi_0/(\mu_1+\mu_2)d$.

5.35. (a) Since the magnetic material is a plane slab and since the permeability is a function of z only, there is no secondary field outside the magnetic material. Hence, $\underset{\sim}{H}_0 = \frac{1}{\mu_0}\underset{\sim}{B}_0 = \frac{B_0}{\mu_0}\underset{\sim}{i}_y$.

(b) Since there are no true currents in the magnetic material, $\underset{\sim}{H}_i = \underset{\sim}{H}_a = \frac{B_0}{\mu_0}\underset{\sim}{i}_y$.

(c) $\underset{\sim}{B}_i = \mu \underset{\sim}{H}_i$

(d) $\underset{\sim}{M} = \dfrac{\underset{\sim}{B}_i}{\mu_0} - \underset{\sim}{H}_i$

(e) $\underset{\sim}{J}_{ms} = \underset{\sim}{M} \times \underset{\sim}{i}_n = [\underset{\sim}{M}]_{z=0} \times (-\underset{\sim}{i}_z)$ for $z=0$, $[\underset{\sim}{M}]_{z=d} \times \underset{\sim}{i}_z$ for $z=d$.

(f) $\underset{\sim}{J}_m = \underset{\sim}{\nabla} \times \underset{\sim}{M}$

For answers to (c), (d), (e), and (f), see page 536 of the text.

5.36. Since $\underset{\sim}{B}$ for $a<r<b$ is independent of θ and ϕ, the permeability of the magnetic material must be a function of r only. Hence

(a) $\underset{\sim}{H} = \dfrac{I}{2\pi r}\underset{\sim}{i}_\phi$ for all r , $\underset{\sim}{B} = \dfrac{\mu I}{2\pi r}\underset{\sim}{i}_\phi$ for $a<r<b$ which upon comparison with the given field yields $\mu = \mu_0 \frac{r}{a}$.

(b) $\underset{\sim}{M} = \dfrac{\underset{\sim}{B}}{\mu_0} - \underset{\sim}{H} = \dfrac{I}{2\pi}\left(\dfrac{1}{a} - \dfrac{1}{r}\right)\underset{\sim}{i}_\phi$.

$\underset{\sim}{J}_{ms} = \underset{\sim}{M} \times \underset{\sim}{i}_n = 0$ for $r=a$, $-\dfrac{I}{2\pi}\left(\dfrac{1}{a} - \dfrac{1}{b}\right)\underset{\sim}{i}_z$ for $r=b$

(c) $\underset{\sim}{J}_m = \underset{\sim}{\nabla} \times \underset{\sim}{M} = \dfrac{I}{2\pi a r}\underset{\sim}{i}_z$.

5.37. $\mu_r = \dfrac{\mu}{\mu_0} = \dfrac{B/H}{\mu_0} = \dfrac{\mu_0 k H^2}{H\mu_0} = kH$; $\mu_{ir} = \dfrac{1}{\mu_0}\dfrac{\Delta B}{\Delta H} = \dfrac{1}{\mu_0}\dfrac{\mu_0 k\, 2H\,\Delta H}{\Delta H} = 2kH$;

$\chi_m = \mu_r - 1 = kH - 1$; $\underset{\sim}{M} = \chi_m \underset{\sim}{H} = (kH-1)\underset{\sim}{H}$.

5.38. See derivations at the ends of Sections 4.3 and 4.5 of the text.

5.39. Let the conducting sheets be in the $z=0$ and $z=d$ planes. Then

$\mathbf{D} = -\rho_{s0}\,\mathbf{i}_z$ for $0<z<d$.

(a) $\mathbf{E} = -\dfrac{\rho_{s0}}{\epsilon_0}\mathbf{i}_z$, $w_e = \frac{1}{2}\mathbf{D}\cdot\mathbf{E} = \dfrac{1}{2\epsilon_0}\rho_{s0}^2$, $W_e = \dfrac{1}{2\epsilon_0}\rho_{s0}^2 d$.

(b) $\mathbf{E} = -\dfrac{\rho_{s0}}{4\epsilon_0}\mathbf{i}_z$, $w_e = \frac{1}{2}\mathbf{D}\cdot\mathbf{E} = \dfrac{1}{8\epsilon_0}\rho_{s0}^2$, $W_e = \dfrac{1}{8\epsilon_0}\rho_{s0}^2 d$.

5.40. Let the conducting sheets be in the $z=0$ and $z=d$ planes and let the uniform electric field be $-E_0\,\mathbf{i}_z$ for $0<z<d$.

(a) $\mathbf{D} = -\epsilon_0 E_0\,\mathbf{i}_z$, $w_e = \frac{1}{2}\mathbf{D}\cdot\mathbf{E} = \frac{1}{2}\epsilon_0 E_0^2$, $W_e = \frac{1}{2}\epsilon_0 E_0^2 d$.

(b) $\mathbf{D} = -4\epsilon_0 E_0\,\mathbf{i}_z$, $w_e = \frac{1}{2}\mathbf{D}\cdot\mathbf{E} = 2\epsilon_0 E_0^2$, $W_e = 2\epsilon_0 E_0^2 d$.

5.41. Let the conducting sheets be in the $z=0$ and $z=d$ planes with the current densities given by $J_{s0}\,\mathbf{i}_x$ and $-J_{s0}\,\mathbf{i}_x$ respectively. Then

$\mathbf{H} = J_{s0}\,\mathbf{i}_y$ for $0<z<d$.

(a) $\mathbf{B} = \mu_0 J_{s0}\,\mathbf{i}_y$, $w_m = \frac{1}{2}\mathbf{H}\cdot\mathbf{B} = \frac{1}{2}\mu_0 J_{s0}^2$, $W_m = \frac{1}{2}\mu_0 J_{s0}^2 d$

(b) $\mathbf{B} = 4\mu_0 J_{s0}\,\mathbf{i}_y$, $w_m = \frac{1}{2}\mathbf{H}\cdot\mathbf{B} = 2\mu_0 J_{s0}^2$, $W_m = 2\mu_0 J_{s0}^2 d$

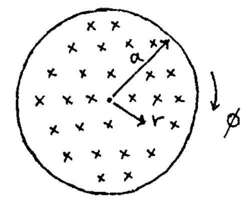
5.42. Let the conducting sheets be in the $z=0$ and $z=d$ planes and let the uniform magnetic field be $B_0\,\mathbf{i}_x$ for $0<z<d$.

(a) $\mathbf{H} = \dfrac{B_0}{\mu_0}\mathbf{i}_x$, $w_m = \frac{1}{2}\mathbf{H}\cdot\mathbf{B} = \dfrac{1}{2}\dfrac{B_0^2}{\mu_0}$, $W_m = \dfrac{1}{2}\dfrac{B_0^2 d}{\mu_0}$

(b) $\mathbf{H} = \dfrac{B_0}{4\mu_0}\mathbf{i}_x$, $w_m = \frac{1}{2}\mathbf{H}\cdot\mathbf{B} = \dfrac{1}{8}\dfrac{B_0^2}{\mu_0}$, $W_m = \dfrac{1}{8}\dfrac{B_0^2 d}{\mu_0}$.

5.43. $W_m = \displaystyle\int_0^{B_0}\mathbf{H}\cdot d\mathbf{B} = \int_0^{H_0} 2\mu_0 k H^2\,dH = \frac{2}{3}\frac{1}{\sqrt{\mu_0 k}}B_0^{3/2}$.

5.44. Applying Faraday's law to a circular path of radius r, we have



$2\pi r E_\phi = -\dfrac{d}{dt}[\mu H_0 \cos\omega t \cdot \pi r^2]$ for $0<r<a$

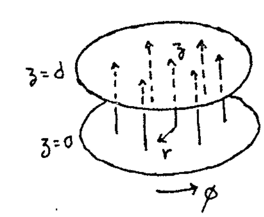
or $\mathbf{E} = -\dfrac{r}{2}\dfrac{d}{dt}[\mu H_0\cos\omega t]\,\mathbf{i}_\phi$

$[\mathbf{P}]_{r=a} = [\mathbf{E}\times\mathbf{H}]_{r=a} = -\dfrac{a}{2}H_0\cos\omega t \cdot \dfrac{d}{dt}(\mu H_0\cos\omega t)\,\mathbf{i}_r$

$\displaystyle\int_S \mathbf{P}\cdot d\mathbf{S} = \int_{\phi=0}^{2\pi}\int_{z=z}^{z+l}[\mathbf{P}]_{r=a}\cdot[-\mathbf{i}_r]\,a\,d\phi\,dz = \frac{d}{dt}\left[\frac{1}{2}\mu H_0^2\cos^2\omega t \cdot \pi a^2\right]$

which is the same as the time rate of change of energy stored in the magnetic field per length l of the material.

5.45. Applying the integral form of Maxwell's curl equation for $\underset{\sim}{H}$ to a circular path of radius r, we have



$$2\pi r H_\phi = 0 + \frac{d}{dt}[\epsilon E_0 \cos\omega t \cdot \pi r^2] \quad \text{for } 0<r<a$$

$$\text{or } \underset{\sim}{H} = \frac{a}{2}\frac{d}{dt}[\epsilon E_0 \cos\omega t]\ \underset{\sim}{i}_\phi$$

$$[\underset{\sim}{P}]_{r=a} = [\underset{\sim}{E}\times\underset{\sim}{H}]_{r=a} = -\frac{a}{2}E_0\cos\omega t \cdot \frac{d}{dt}(\epsilon E_0\cos\omega t)\ \underset{\sim}{i}_r$$

$$\int_S \underset{\sim}{P}\cdot d\underset{\sim}{S} = \int_{\phi=0}^{2\pi}\int_{z=0}^{d}[\underset{\sim}{P}]_{r=a}\cdot[-\underset{\sim}{i}_r]\ a\, d\phi\, dz = \frac{d}{dt}\left[\frac{1}{2}\epsilon E_0^2\cos^2\omega t \cdot \pi a^2 d\right]$$

which is the same as the time rate of change of energy stored in the electric field in the material.

5.46. $[E_{\theta 2}]_{r=a} = [E_{\theta 1}]_{r=a} = E_0\, \underset{\sim}{i}_z\cdot\underset{\sim}{i}_\theta = -E_0\sin\theta$

$[E_{\phi 2}]_{r=a} = [E_{\phi 1}]_{r=a} = E_0\,\underset{\sim}{i}_z\cdot\underset{\sim}{i}_\phi = 0$

$[D_{r2}]_{r=a} = [D_{r1}]_{r=a} = 2\epsilon_0 E_0\,\underset{\sim}{i}_z\cdot\underset{\sim}{i}_r = 2\epsilon_0 E_0\cos\theta$

$[E_{r2}]_{r=a} = \frac{1}{\epsilon_2}[D_{r2}]_{r=a} = \frac{1}{2}E_0\cos\theta.$

Thus $[\underset{\sim}{E}_2]_{r=a} = \frac{E_0}{2}\cos\theta\ \underset{\sim}{i}_r - E_0\sin\theta\ \underset{\sim}{i}_\theta.$

5.47. $B_{z2} = B_{z1} = 5B_0.$

From $\underset{\sim}{i}_n\times(\underset{\sim}{H}_1-\underset{\sim}{H}_2) = \underset{\sim}{J}_S$, we have

$H_{y1}-H_{y2} = -J_{Sx}$ or $H_{y2} = H_{y1}+J_{Sx} = 2\frac{B_0}{\mu_0}$, $B_{y2} = 4B_0$

$H_{x1}-H_{x2} = J_{Sy}$ or $H_{x2} = H_{x1}-J_{Sy} = 5\frac{B_0}{2\mu_0}$, $B_{x2} = 5B_0$

Thus $\underset{\sim}{B}_2 = B_0(5\,\underset{\sim}{i}_x + 4\,\underset{\sim}{i}_y + 5\,\underset{\sim}{i}_z).$

5.48. (a) $-\frac{\partial \underset{\sim}{B}}{\partial t} = \underset{\sim}{\nabla}\times\underset{\sim}{E} = -E_0\frac{\pi}{a}\cos\frac{\pi x}{a}\cos\frac{\pi}{a\sqrt{\mu_0\epsilon_0}}t\ \underset{\sim}{i}_y$

$$\underset{\sim}{H} = \frac{\underset{\sim}{B}}{\mu_0} = E_0\sqrt{\frac{\epsilon_0}{\mu_0}}\cos\frac{\pi x}{a}\sin\frac{\pi}{a\sqrt{\mu_0\epsilon_0}}t\ \underset{\sim}{i}_y$$

(b) $[\underset{\sim}{J}_S]_{x=0} = \underset{\sim}{i}_x\times[\underset{\sim}{H}]_{x=0} = E_0\sqrt{\frac{\epsilon_0}{\mu_0}}\sin\frac{\pi}{a\sqrt{\mu_0\epsilon_0}}t\ \underset{\sim}{i}_z$

$[J_S]_{x=a} = -\underset{\sim}{i}_x\times[\underset{\sim}{H}]_{x=a} = E_0\sqrt{\frac{\epsilon_0}{\mu_0}}\sin\frac{\pi}{a\sqrt{\mu_0\epsilon_0}}\ \underset{\sim}{i}_z.$

5.49. (a) Using $-\frac{\partial\underset{\sim}{B}}{\partial t} = \underset{\sim}{\nabla}\times\underset{\sim}{E}$ and $\underset{\sim}{H} = \frac{\underset{\sim}{B}}{\mu}$, we obtain the expressions for $\underset{\sim}{H}_1$ and $\underset{\sim}{H}_2$ given on page 536 of the text.

(b) From continuity of the tangential component of $\underset{\sim}{E}$, we have

$$[E_{x1}]_{z=0} = [E_{x2}]_{z=0} \quad \text{or} \quad E_i + E_r = E_t.$$

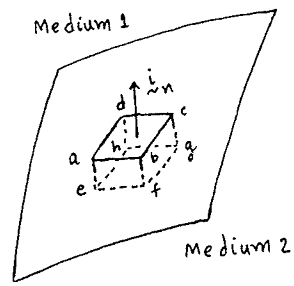
The tangential component of $\underset{\sim}{H}$ is also continuous here because there is no true surface current at the boundary. Thus

$$[H_{y1}]_{z=0} = [H_{y2}]_{z=0} \quad \text{or} \quad E_i - E_r = 2E_t.$$

Solving the two equations, we obtain $\frac{E_r}{E_i} = -\frac{1}{3}$ and $\frac{E_t}{E_i} = \frac{2}{3}$.

5.50. Considering an infinitesimal rectangular box abcdefgh symmetrically situated about the boundary between two media, we have, as the side surfaces tend to zero,

$\oint_{abcda} \underset{\sim}{E}_1 \cdot d\underset{\sim}{l} = \oint_{efghe} \underset{\sim}{E}_2 \cdot d\underset{\sim}{l}$ from the boundary condition for the tangential component of $\underset{\sim}{E}$. But from Faraday's law,



$$\oint_{abcda} \underset{\sim}{E}_1 \cdot d\underset{\sim}{l} = -\frac{d}{dt}\int_{abcd} \underset{\sim}{B}_1 \cdot d\underset{\sim}{S} = -\frac{d}{dt}[(\underset{\sim}{i}_n \cdot \underset{\sim}{B}_1)(abcd)]$$

$$\oint_{efghe} \underset{\sim}{E}_2 \cdot d\underset{\sim}{l} = -\frac{d}{dt}\int_{efgh} \underset{\sim}{B}_2 \cdot d\underset{\sim}{S} = -\frac{d}{dt}[(\underset{\sim}{i}_n \cdot \underset{\sim}{B}_2)(efgh)]$$

$$\therefore \frac{d}{dt}\{[\underset{\sim}{i}_n \cdot (\underset{\sim}{B}_1 - \underset{\sim}{B}_2)](abcd)\} = 0$$

or, $\underset{\sim}{i}_n \cdot (\underset{\sim}{B}_1 - \underset{\sim}{B}_2) =$ constant with time.

But, for the particular case of sinusoidal steady state,

$$\frac{d}{dt}\{[\underset{\sim}{i}_n \cdot (\underset{\sim}{B}_1 - \underset{\sim}{B}_2)](abcd)\} \rightarrow [j\omega\, \underset{\sim}{i}_n \cdot (\underset{\sim}{B}_1 - \underset{\sim}{B}_2)](abcd) = 0$$

$$\therefore \underset{\sim}{i}_n \cdot (\underset{\sim}{B}_1 - \underset{\sim}{B}_2) = 0 \quad \text{for all time.}$$

Considering an infinitesimal rectangular box abcdefgh symmetrically situated about the boundary between two media, we have, as the side surfaces tend to zero,

$$\oint_{abcda} \underset{\sim}{H}_1 \cdot d\underset{\sim}{l} - \oint_{efghe} \underset{\sim}{H}_2 \cdot d\underset{\sim}{l} = \oint_{abcda} \underset{\sim}{J}_s \cdot \underset{\sim}{i}_n \times d\underset{\sim}{l} = -\int_{abcd} (\underset{\sim}{\nabla}_s \cdot \underset{\sim}{J}_s)\, dS$$

$= -(\underset{\sim}{\nabla}_s \cdot \underset{\sim}{J}_s)(abcd)$ from the boundary condition for the tangential component of $\underset{\sim}{H}$. But from the integral form of Maxwell's curl equation for $\underset{\sim}{H}$,

$$\oint_{abcda} \underset{\sim}{H}_1 \cdot d\underset{\sim}{l} = \int_{abcd} \underset{\sim}{J}_1 \cdot d\underset{\sim}{S} + \frac{d}{dt}\int_{abcd} \underset{\sim}{D}_1 \cdot d\underset{\sim}{S} = \left[\underset{\sim}{i}_n \cdot \underset{\sim}{J}_1 + \frac{d}{dt}(\underset{\sim}{i}_n \cdot \underset{\sim}{D}_1)\right](abcd)$$

$$\oint_{efghe} \underset{\sim}{H}_2 \cdot d\underset{\sim}{l} = \int_{efgh} \underset{\sim}{J}_2 \cdot d\underset{\sim}{s} + \frac{d}{dt}\int_{efgh} \underset{\sim}{D}_2 \cdot d\underset{\sim}{s} = \left[\underset{\sim}{i}_n \cdot \underset{\sim}{J}_2 + \frac{d}{dt}(\underset{\sim}{i}_n \cdot \underset{\sim}{D}_2)\right](efgh)$$

$$\therefore \left\{ \underset{\sim}{i}_n \cdot (\underset{\sim}{J}_1 - \underset{\sim}{J}_2) + \frac{d}{dt}[\underset{\sim}{i}_n \cdot (\underset{\sim}{D}_1 - \underset{\sim}{D}_2)]\right\}(abcd) = -(\underset{\sim}{\nabla}_s \cdot \underset{\sim}{J}_s)(abcd).$$

But, from the boundary condition for the normal component of $\underset{\sim}{J}$,

$$[\underset{\sim}{i}_n \cdot (\underset{\sim}{J}_1 - \underset{\sim}{J}_2)](abcd) = -(\underset{\sim}{\nabla}_s \cdot \underset{\sim}{J}_s)(abcd) - \frac{d}{dt}[\rho_s\,(abcd)].$$

$$\therefore \frac{d}{dt}\left\{[\underset{\sim}{i}_n \cdot (\underset{\sim}{D}_1 - \underset{\sim}{D}_2)](abcd) - \rho_s(abcd)\right\} = 0$$

or, $\underset{\sim}{i}_n \cdot (\underset{\sim}{D}_1 - \underset{\sim}{D}_2) - \rho_s$ = constant with time.

But, for the particular case of sinusoidal steady state,

$$\frac{d}{dt}\left\{[\underset{\sim}{i}_n \cdot (\underset{\sim}{D}_1 - \underset{\sim}{D}_2)](abcd) - \rho_s(abcd)\right\} \rightarrow j\omega[\underset{\sim}{i}_n \cdot (\underset{\sim}{D}_1 - \underset{\sim}{D}_2) - \rho_s](abcd) = 0$$

$\therefore$ $\underset{\sim}{i}_n \cdot (\underset{\sim}{D}_1 - \underset{\sim}{D}_2) - \rho_s = 0$ for all time

or $\underset{\sim}{i}_n \cdot (\underset{\sim}{D}_1 - \underset{\sim}{D}_2) = \rho_s$.

CHAPTER 6

6.1. Method of solution is similar to that of Example 6-1. For answers, see page 536 of the text.

6.2. The equation of motion of the electron is given by

$$m \frac{d^2x}{dt^2} \underset{\sim}{i}_x = e E_x \underset{\sim}{i}_x \quad \text{or} \quad m \frac{d^3x}{dt^3} = e \frac{\partial E_x}{\partial x} \frac{dx}{dt}$$

But $\frac{\partial E_x}{\partial x} = \underset{\sim}{\nabla} \cdot \underset{\sim}{E} = \frac{\rho}{\epsilon_0}$ where ρ is the charge density between the plates.

Thus $m \frac{d^3x}{dt^3} = e \frac{\rho}{\epsilon_0} v(x) = \frac{e}{\epsilon_0} J_0$ or $\frac{d^3x}{dt^3} = \frac{e J_0}{m \epsilon_0}$.

Integrating three times and evaluating the arbitrary constants by using $x=0$, $\frac{dx}{dt}=0$, and $\frac{d^2x}{dt^2}=0$ for $t=0$, we get

$$x = \frac{e J_0}{m\epsilon_0} \frac{t^3}{6} \quad \text{and} \quad v = \frac{e J_0}{m \epsilon_0} \frac{t^2}{2}.$$

From $|e| V = \frac{1}{2} m v^2$, we then obtain

$$V = \frac{|e| J_0^2 t^4}{8 m \epsilon_0^2} = \frac{|e| J_0^2}{8 m \epsilon_0^2} \left[\frac{6 m \epsilon_0 x}{|e|(-J_0)} \right]^{4/3}$$

$$= \frac{1}{8} \left[-\sqrt{\frac{m}{|e|}} \frac{J_0}{\epsilon_0} \right]^{2/3} (x)^{4/3} 6^{4/3} = \left[\frac{3}{2} \sqrt{k}\, d \right]^{4/3} \left(\frac{x}{d} \right)^{4/3} = V_0 \left(\frac{x}{d} \right)^{4/3}$$

which agrees with Equation 6-22 of the text.

6.3. Verification consists of solving the one-dimensional Laplace's equations to obtain the general solutions and then substituting the boundary conditions to obtain the particular solutions.

6.4. Method similar to that of Example 6-4.

$$V = \begin{cases} V_0 \dfrac{\epsilon_1 \ln c/b - \epsilon_2 \ln c/r}{\epsilon_1 \ln c/b - \epsilon_2 \ln c/a} & \text{for } a<r<c \\[2ex] V_0 \dfrac{\epsilon_1 \ln r/b}{\epsilon_1 \ln c/b - \epsilon_2 \ln c/a} & \text{for } c<r<b \end{cases}$$

$$[V]_{r=c} = V_0 \frac{\epsilon_1 \ln c/b}{\epsilon_1 \ln c/b - \epsilon_2 \ln c/a}.$$

6.5. Since ϵ is a function of x, Laplace's equation is given by

$$\nabla \cdot \epsilon \nabla V = \nabla V \cdot \nabla \epsilon + \epsilon \nabla \cdot \nabla V = 0$$

or $$\left(\frac{dV}{dx}\right)\left(\frac{d\epsilon}{dx}\right) + \epsilon \frac{d^2V}{dx^2} = 0$$

For $\epsilon = \epsilon_1 + (\epsilon_2 - \epsilon_1)\frac{x}{d}$; we have

$$\frac{\epsilon_2 - \epsilon_1}{d}\frac{dV}{dx} + \left[\epsilon_1 + (\epsilon_2 - \epsilon_1)\frac{x}{d}\right]\frac{d^2V}{dx^2} = 0$$

$$\frac{d}{dx}\left(\frac{dV}{dx}\right) = -\frac{\epsilon_2 - \epsilon_1}{\epsilon_1 d + (\epsilon_2 - \epsilon_1)x}\frac{dV}{dx}$$

$$\frac{d\left(\frac{dV}{dx}\right)}{\frac{dV}{dx}} = -\frac{\epsilon_2 - \epsilon_1}{\epsilon_1 d + (\epsilon_2 - \epsilon_1)x}\,dx$$

Solving for V using the boundary conditions $V = 0$ for $x = 0$ and $V = V_0$ for $x = d$, we obtain

$$V = \frac{V_0}{\ln \frac{\epsilon_2}{\epsilon_1}} \ln \frac{\epsilon_1 d + (\epsilon_2 - \epsilon_1)x}{\epsilon_1 d}$$

$$\mathbf{E} = -\frac{\partial V}{\partial x}\mathbf{i}_x = -\frac{V_0}{\ln \frac{\epsilon_2}{\epsilon_1}}\frac{(\epsilon_2 - \epsilon_1)}{\epsilon_1 d + (\epsilon_2 - \epsilon_1)x}\mathbf{i}_x$$

6.6. $\nabla \cdot \mathbf{M} = -M_0$ for $0 < x < d$, and $(\mathbf{M}_2 - \mathbf{M}_1)\cdot \mathbf{i}_n = M_0 d$ for $x = 0$ and 0 for $x = d$.

Hence, the analogous charge distribution is

$\rho = \epsilon M_0$ for $0 < x < d$, and $\rho_S = -\epsilon M_0 d$ for $x = 0$ and 0 for $x = d$.

The electric field intensity due to this charge distribution is

$\mathbf{E} = M_0(x - d)\,\mathbf{i}_x$ for $0 < x < d$ and 0 otherwise.

Thus $\mathbf{H} = M_0(x - d)\,\mathbf{i}_x$ for $0 < x < d$ and 0 otherwise

$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = 0$ everywhere.

6.7. (a) $\nabla \cdot \mathbf{M} = 0$ and $(\mathbf{M}_2 - \mathbf{M}_1)\cdot \mathbf{i}_n = (0 - M_0\mathbf{i}_z)\cdot \mathbf{i}_r = -M_0 \cos\theta$ for $r = a$.

Hence, the analogous charge distribution is

$\rho = 0$ and $\rho_S = \epsilon M_0 \cos\theta$ for $r = a$.

(b) The answer to part (d) of Problem 5.11 gives the electric field intensity due to a surface charge of density $3\epsilon_0 E_0 \cos\theta$ on the surface $r = a$. Hence, the required field is as given on page 537 of the text.

(c) See page 537 of the text.

6.8. The only general solution that allows the substitution of the given boundary conditions is that corresponding to $\alpha = 0$ in Eq.(6-38), that is, $V = (A_0 x + B_0)(C_0 x + d)$. Substituting the given boundary conditions, we obtain $V = 50\,xy$ which then gives

$$\mathbf{E} = -\nabla V = -50\,y\,\mathbf{i}_x - 50\,x\,\mathbf{i}_y$$

$$[\rho_s]_{x=0,\,y>0} = [\epsilon_0 E_x]_{x=0,\,y>0} = -50\,\epsilon_0\,y$$

$$[\rho_s]_{y=0,\,x>0} = [\epsilon_0 E_y]_{y=0,\,x>0} = -50\,\epsilon_0\,x$$

$$[\rho_s]_{xy=2} = [\mathbf{E}]_{xy=2}\cdot[\mathbf{i}_n]_{xy=2} = \left[(-50y\,\mathbf{i}_x - 50x\,\mathbf{i}_y)\cdot\left(\frac{-y\,\mathbf{i}_x - x\,\mathbf{i}_y}{\sqrt{x^2+y^2}}\right)\right]_{xy=2}$$

$$= \frac{50}{x}\sqrt{x^4+4}\,.$$

6.9. This problem is similar to Example 6-6 except for the boundary condition for $x = a$, $0<y<b$. Following the method of Example 6-6, we therefore have, for $V = V_1 \sin\frac{\pi y}{b} + V_2 \sin\frac{3\pi y}{b}$ for $x = a$, $0<y<b$,

$$V_1 \sin\frac{\pi y}{b} + V_2 \sin\frac{3\pi y}{b} = \sum_{n=1,2,3,\cdots}^{\infty} A_n' \sinh\frac{n\pi a}{b}\sin\frac{n\pi y}{b} \quad \text{for } 0<y<b$$

which gives $A_1' = \dfrac{V_1}{\sinh\frac{\pi a}{b}}$, $A_3' = \dfrac{V_2}{\sinh\frac{3\pi a}{b}}$, and $A_n' = 0$ for all other n,

resulting in the solution for V given on page 537 of the text.

Similarly, for $V = V_1 \sin^3\frac{\pi y}{b}$ for $x = a$, $0<y<b$, the answer given on page 537 of the text can be obtained by recognizing that

$$\sin^3\frac{\pi y}{b} = \frac{3}{4}\sin\frac{\pi y}{b} - \frac{1}{4}\sin\frac{3\pi y}{b}.$$

6.10. The boundary conditions are

$V = 0$ for $x = 0$, $-\infty<y<\infty$

$V = 0$ for $x = a$, $-\infty<y<\infty$

$V = V_0$ for $y = 0$, $0<x<a$

Since V is required to be zero for two values of x in the range $-\infty<y<\infty$, the only general solution we need to consider is that corresponding to $\alpha \neq 0$ in Eq.(6-38). Using the first two boundary conditions, we then obtain

$$V(x,y) = \sum_{n=1,2,3,\cdots}^{\infty}\left(C_n e^{\frac{n\pi y}{a}} + D_n e^{-\frac{n\pi y}{a}}\right)\sin\frac{n\pi x}{a}$$

We now make use of an additional (unspecified) condition that $V=0$ at $y=\pm\infty$ to set $C_n=0$ for $y>0$ and $D_n=0$ for $y<0$. Finally, making use of the third boundary condition, we get the required solution as

$$V = \begin{cases} \sum_{n=1,3,5,\cdots}^{\infty} \frac{4V_0}{n\pi} e^{-\frac{n\pi y}{a}} \sin\frac{n\pi x}{a} & \text{for } y>0 \\ \sum_{n=1,3,5,\cdots}^{\infty} \frac{4V_0}{n\pi} e^{\frac{n\pi y}{a}} \sin\frac{n\pi x}{a} & \text{for } y<0 \end{cases}$$

For large values of y, the most significant term is $e^{-\frac{\pi|y|}{a}}$. Hence, for large values of y, $V \approx \frac{4V_0}{\pi} e^{-\frac{\pi|y|}{a}} \sin\frac{\pi x}{a}$.

6.11. (a) Using the methods of Examples 6-7 and 6-8 and recognizing that V must be an odd function of $(x-\frac{a}{2})$, we get the answers given on page 537 of the text.

(b) Using the methods of Examples 6-7 and 6-8 and recognizing that V must be an even function of $(x-\frac{a}{2})$, we get the answers given on page 537 of the text.

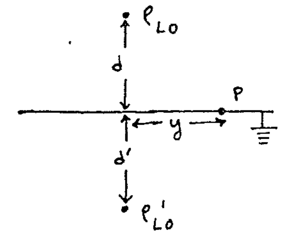
(c) We use superposition for this case by considering two different sets of boundary conditions as follows:

$V_I=0$	for $y=0,\ 0<x<a$	$V_{II}=0$	for $y=0,\ 0<x<a$
$V_I=0$	for $x=0,\ 0<y<b$	$V_{II}=0$	for $x=0,\ 0<y<b$
$V_I=V_1$	for $x=a,\ 0<y<b$	$V_{II}=0$	for $x=a,\ 0<y<b$
$V_I=0$	for $y=b,\ 0<x<a$	$V_{II}=V_2$	for $y=b,\ 0<x<a$

The solution for V_I is given in Example 6-7 and the solution for V_{II} can be written by inspection. The required solution is then given by $V=V_I+V_{II}$ since V_I+V_{II} satisfies the given boundary conditions and also $\nabla^2 V = \nabla^2 V_I + \nabla^2 V_{II} = 0$. For answers, see page 537 of the text.

6.12. 71.5, 42.9, and 42.9 volts.

6.13. Let us postulate an infinitely long image line charge of uniform density $\rho'_{\ell 0}$ at a distance d' from the grounded conductor and directly beneath the actual line charge. Then the expression for the potential at point P on the conductor surface is given by



$$V = -\frac{\rho_{L0}}{2\pi\epsilon_0}\ln\frac{\sqrt{y^2+d^2}}{d} - \frac{\rho'_{L0}}{2\pi\epsilon_0}\ln\frac{\sqrt{y^2+d'^2}}{d'}$$

setting this quantity equal to zero, we obtain

$\rho'_{L0} = -\rho_{L0}$ and $d' = d$. Then, the induced surface charge density is given by

$$\rho_S = \mathbf{D}\cdot\mathbf{i}_n = -\frac{\rho_{L0}\,d}{\pi(y^2+d^2)}$$ which gives $-\rho_{L0}$ for the induced surface charge per unit length parallel to the line charge.

6.14. (a) $-Q$ at $(a,-b)$, Q at $(-a,b)$, and $-Q$ at $(-a,-b)$.

$$\mathbf{E} = \begin{cases} -\dfrac{Qb}{2\pi\epsilon_0}\left\{\dfrac{1}{[(x-a)^2+b^2+z^2]^{3/2}} - \dfrac{1}{[(x+a)^2+b^2+z^2]^{3/2}}\right\}\mathbf{i}_y & \text{for } y=0,\ x>0 \\ -\dfrac{Qa}{2\pi\epsilon_0}\left\{\dfrac{1}{[a^2+(y-b)^2+z^2]^{3/2}} - \dfrac{1}{[a^2+(y+b)^2+z^2]^{3/2}}\right\}\mathbf{i}_x & \text{for } x=0,\ y>0 \end{cases}$$

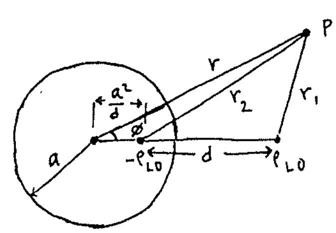
$$\rho_S = \begin{cases} \dfrac{Qb}{2\pi}\left\{\dfrac{1}{[(x+a)^2+b^2+z^2]^{3/2}} - \dfrac{1}{[(x-a)^2+b^2+z^2]^{3/2}}\right\} & \text{for } y=0,\ x>0 \\ \dfrac{Qa}{2\pi}\left\{\dfrac{1}{[a^2+(y+b)^2+z^2]^{3/2}} - \dfrac{1}{[a^2+(y-b)^2+z^2]^{3/2}}\right\} & \text{for } x=0,\ y>0 \end{cases}$$

Total induced charge

$$= \int_{x=0}^{\infty}\int_{z=-\infty}^{\infty}[\rho_S]_{y=0,\,x>0}\,dz\,dx + \int_{y=0}^{\infty}\int_{z=-\infty}^{\infty}[\rho_S]_{x=0,\,y>0}\,dz\,dy = -Q \text{ C}.$$

(b) $-Q$ at $(-a,\sqrt{3}a)$, Q at $(-2a,0)$, $-Q$ at $(-a,-\sqrt{3}a)$, Q at $(a,-\sqrt{3}a)$, and $-Q$ at $(2a,0)$.

6.15. The potential at point P due to the actual and the image line charges is



$$V = -\frac{\rho_{L0}}{2\pi\epsilon_0}\ln\frac{r_1}{d-a} + \frac{\rho_{L0}}{2\pi\epsilon_0}\ln\frac{r_2}{a-\frac{a^2}{d}}$$

which gives

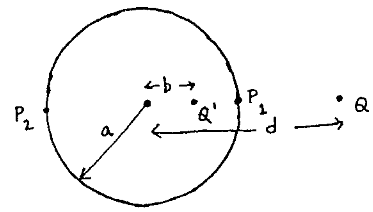
$$\mathbf{E} = -\nabla V = -\nabla\left(\frac{\rho_{L0}}{2\pi\epsilon_0}\ln\frac{r_2}{r_1}\right)$$

$$= -\nabla\,\frac{\rho_{L0}}{2\pi\epsilon_0}\ln\frac{[r^2+(a^2/d)^2-2r\frac{a^2}{d}\cos\phi]^{1/2}}{[r^2+d^2-2rd\cos\phi]^{1/2}}$$

Evaluating $[\underset{\sim}{E}]_{r=a}$ and using $\rho_s = \epsilon_0 [E_r]_{r=a}$, we obtain

$$\rho_s = -\frac{\rho_{L0}}{2\pi a} \frac{d^2 - a^2}{d^2 + a^2 - 2ad\cos\phi} \quad \text{and} \quad \int_{\phi=0}^{2\pi} \rho_s \, a \, d\phi = -\rho_{L0}.$$

6.16. Let us assume the image charge to be a point charge of Q' coulombs situated as shown in the figure. Then



$$[V]_{P_1} = \frac{Q}{4\pi\epsilon_0(d-a)} + \frac{Q'}{4\pi\epsilon_0(a-b)} = 0$$

$$[V]_{P_2} = \frac{Q}{4\pi\epsilon_0(d+a)} + \frac{Q'}{4\pi\epsilon_0(a+b)} = 0$$

Solving these two equations, we get $Q' = -Q\frac{a}{d}$ and $b = \frac{a^2}{d}$. The induced charge on the conductor surface must be equal to the image charge value, that is, $-Q\frac{a}{d}$ coulombs since the field lines which would end on the image charge in the absence of the conductor would end on the conductor. This result can be also be deduced mathematically by finding the surface charge density on the conductor surface and evaluating its integral over the surface.

6.17. (a) $\underset{\sim}{E} = -\underset{\sim}{\nabla} V$; see page 537 of the text for answer.

(b) $$\rho_s = \underset{\sim}{D} \cdot \underset{\sim}{i}_n = \begin{cases} \epsilon_1 [\underset{\sim}{E}]_{x=0} \cdot \underset{\sim}{i}_x & \text{for } x = 0 \\ \epsilon_2 [\underset{\sim}{E}]_{x=d} \cdot (-\underset{\sim}{i}_x) & \text{for } x = d \end{cases}$$

See page 537 of the text for answer.

(c) $C = \dfrac{\text{charge per unit area}}{V_0} = \dfrac{\rho_s}{V_0} = \dfrac{\epsilon_1\epsilon_2}{\epsilon_2 t + \epsilon_1(d-t)}$ which can be written as given in the problem.

6.18. (a) Using the result for $\underset{\sim}{E}$ given in Problem 6.5, we obtain

$$\underset{\sim}{D} = \epsilon \underset{\sim}{E} = -\frac{(\epsilon_2 - \epsilon_1) V_0}{d \ln \epsilon_2/\epsilon_1} \underset{\sim}{i}_x$$

$$\rho_s = \underset{\sim}{D} \cdot \underset{\sim}{i}_n = \begin{cases} -(\epsilon_2 - \epsilon_1) V_0 / d \ln \frac{\epsilon_2}{\epsilon_1} & \text{for } x = 0 \\ (\epsilon_2 - \epsilon_1) V_0 / d \ln \frac{\epsilon_2}{\epsilon_1} & \text{for } x = d \end{cases}$$

$$C = \frac{|\rho_s|}{V_0} = \frac{\epsilon_2 - \epsilon_1}{d \ln \frac{\epsilon_2}{\epsilon_1}}.$$

(b) Finding $w_e = \frac{1}{2}\mathbf{D}\cdot\mathbf{E}$ and evaluating $W_e = \int_{x=0}^{d}\int_{y=0}^{1}\int_{z=0}^{1} w_e\,dx\,dy\,dz$,

we obtain the electric stored energy per unit area of the plates as

$\frac{1}{2}\,\frac{V_0^2(\epsilon_2-\epsilon_1)}{d\ln \epsilon_2/\epsilon_1}$ which gives the same expression for C as in (a).

(c) For a slab of infinitesimal thickness dx and located at a distance x from $x=0$, $dc = \dfrac{\epsilon_1 + (\epsilon_2-\epsilon_1)\frac{x}{d}}{dx}$. We then have

$$\frac{1}{c} = \int_{x=0}^{d}\frac{1}{dc} = \int_{x=0}^{d}\frac{dx}{\epsilon_1+(\epsilon_2-\epsilon_1)\frac{x}{d}} = \frac{d}{\epsilon_2-\epsilon_1}\ln\frac{\epsilon_2}{\epsilon_1}.$$

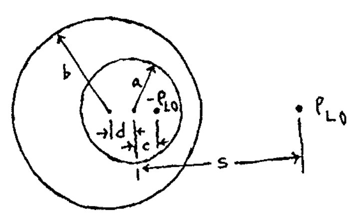
6.19. Let us choose the z axis to be the axis of the conductor system and let the potentials be $V=0$ and $V=V_0$ on the surfaces $r=a$ and $r=b$, respectively. Then, from Table 6.1, the solution for the potential between the conductors is given by $V = V_0\,(\ln\frac{r}{a})/(\ln\frac{b}{a})$. Evaluating $\mathbf{E} = -\nabla V$ and using $\rho_s = \mathbf{D}\cdot\mathbf{i}_n$, we get

$$\rho_s = \begin{cases} -\epsilon V_0/a\ln\frac{b}{a} & \text{for } r=a \\ \epsilon V_0/a\ln\frac{b}{a} & \text{for } r=b \end{cases}$$

The surface charge per unit length of the surface is then given by $\int_{\phi=0}^{2\pi}\rho_s\,a\,d\phi$ for $r=a$ and $\int_{\phi=0}^{2\pi}\rho_s\,b\,d\phi$ for $r=b$. Evaluating these and dividing the magnitude by V_0, we get $\mathcal{C} = \dfrac{2\pi\epsilon}{\ln\frac{b}{a}}$ from which the expressions for $\mathcal{G}$ and $\mathcal{L}$ follow from Eqs. (6-93) and (6-96), respectively.

6.20. Letting the locations of line charges of equal and opposite uniform densities to be as shown in the figure, we have

$a^2 = cs$ and $b^2 = (c+d)(s+d)$.



Solving these two equations, we obtain

$$s = \frac{(b^2-a^2-d^2)+\sqrt{(b^2-a^2-d^2)^2-4a^2d^2}}{2d}$$

$$c = \frac{(b^2-a^2-d^2)-\sqrt{(b^2-a^2-d^2)^2-4a^2d^2}}{2d}$$

The potential difference between the two conductors is

$$V_0 = \frac{\rho_{L0}}{2\pi\epsilon}\ln\frac{s+d+b}{b-c-d} - \frac{\rho_{L0}}{2\pi\epsilon}\ln\frac{s-a}{a-c}$$

Substituting for s and c and simplifying, we obtain

$$V_0 = -\frac{\rho_{L0}}{2\pi\epsilon}\ln\left\{\frac{a^2+b^2-d^2}{2ab}+\sqrt{\left(\frac{a^2+b^2-d^2}{2ab}\right)^2-1}\right\}$$

$$= -\frac{\rho_{L0}}{2\pi\epsilon}\cosh^{-1}\frac{a^2+b^2-d^2}{2ab}.$$

Thus $\mathcal{C} = \dfrac{2\pi\epsilon}{\cosh^{-1}\dfrac{a^2+b^2-d^2}{2ab}}$, $\mathcal{G} = \dfrac{2\pi\sigma}{\cosh^{-1}\dfrac{a^2+b^2-d^2}{2ab}}$, $\mathcal{L} = \dfrac{\mu}{2\pi}\cosh^{-1}\dfrac{a^2+b^2-d^2}{2ab}$.

For $d\to 0$, $\cosh^{-1}\dfrac{a^2+b^2-d^2}{2ab} \to \ln\left\{\dfrac{a^2+b^2}{2ab}+\sqrt{\left(\dfrac{a^2+b^2}{2ab}\right)^2-1}\right\} = \ln\dfrac{b}{a}$

which reduces the expressions for $\mathcal{C}$, $\mathcal{G}$, and $\mathcal{L}$ to those of Figure 6.15 (b).

6.21. Method is similar to that of Example 6-13.

$$\mathbf{B} = \frac{\mu J_0 r^2}{3a}\mathbf{i}_\phi, \quad d\psi_i = \frac{\mu J_0 r^2 l\,dr}{3a}, \quad N = \frac{r^3}{a^3}, \quad N\,d\psi_i = \frac{\mu J_0 l r^5\,dr}{3a^4}.$$

$$\psi_i = \int_{r=0}^{a} N\,d\psi_i = \frac{\mu J_0 l a^2}{18}, \quad \mathcal{L}_i = \frac{\psi_i}{lI} = \frac{\mu J_0 l a^2/18}{2\pi J_0 a^2 l/3} = \frac{\mu}{12\pi}.$$

Alternatively, $W_{mi} = \displaystyle\int_{r=0}^{a}\int_{\phi=0}^{2\pi}\int_{z=0}^{1}\frac{\mu J_0^2 r^4}{18a^2}\,r\,dr\,d\phi\,dz = \frac{\pi\mu J_0^2 a^4}{54}$,

$$\mathcal{L}_i = \frac{2W_{mi}}{I^2} = \frac{\mu}{12\pi}.$$

6.22. Choosing the z axis to be the axis of the toroid and using Ampere's circuital law, we get $\mathbf{B} = \dfrac{\mu a N I}{r}\mathbf{i}_\phi$ inside the toroid. Then

$$\psi = \int_{r=a-\frac{b}{2}}^{a+\frac{b}{2}}\frac{\mu a N I}{r}\,dr\,dz = \mu a N I c\ln\frac{2a+b}{2a-b}.$$

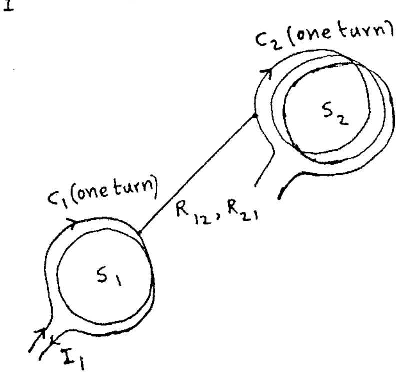
$$L = 2\pi a N\frac{\psi}{I} = 2\pi\mu a^2 N^2 c\ln\frac{2a+b}{2a-b}.$$

6.23. Choosing the z axis to be the axis of the solenoid, we have

$\mathbf{B} = \mu N I\,\mathbf{i}_z$ inside the solenoid. Then

$\psi = B_z\pi a^2 = \pi a^2\mu N I$ and $\mathcal{L} = N\dfrac{\psi}{I} = \pi a^2\mu N^2$.

6.24. Let us consider two windings having N_1 and N_2 turns and with currents I_1 and I_2 respectively. Then, if $\mathbf{B}_1$ and $\mathbf{B}_2$ are the magnetic fields and $\mathbf{A}_1$ and $\mathbf{A}_2$ are the vector potentials due to the N_1-turn winding and the N_2-turn



winding, respectively, we have

$$\Psi_{12} = \int_{S_1} \mathbf{B}_2 \cdot d\mathbf{S}_1 = \int_{S_1} \nabla \times \mathbf{A}_2 \cdot d\mathbf{S}_1 = \oint_{C_1} \mathbf{A}_2 \cdot d\mathbf{l}_1$$

$$= \oint_{C_1} \left[\frac{N_2 \mu I_2}{4\pi} \oint_{C_2} \frac{d\mathbf{l}_2}{R_{12}} \right] \cdot d\mathbf{l}_1 = \frac{\mu N_2 I_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{R_{12}}$$

Similarly, $\Psi_{21} = \dfrac{\mu N_1 I_1}{4\pi} \oint_{C_2} \oint_{C_1} \dfrac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{R_{21}}$

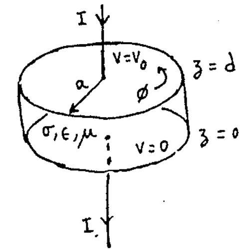
Thus $\dfrac{\Psi_{12}}{N_2 I_2} = \dfrac{\Psi_{21}}{N_1 I_1}$ or $N_1 \dfrac{\Psi_{12}}{I_2} = N_2 \dfrac{\Psi_{21}}{I_1}$ or $L_{12} = L_{21}$.

6.25. The magnetic flux produced by the solenoid of radius b and linking one turn of the solenoid of radius a is $\mu N_2 I_2 (\pi a^2)$. Hence,

$$L_{12} = N_1 \frac{\mu N_2 I_2 \pi a^2}{I_2} = \mu \pi a^2 N_1 N_2 .$$

Alternatively, the magnetic flux produced by the solenoid of radius a and linking one turn of the solenoid of radius b is $\mu N_1 I_1 (\pi a^2) + 0\,[\pi(b^2 - a^2)]$ or $\mu N_1 I_1 \pi a^2$ which gives $L_{21} = N_2 \dfrac{\mu N_1 I_1 \pi a^2}{I_1} = \mu \pi a^2 N_1 N_2 = L_{12}$.

6.26. (a) From symmetry considerations, $\mathbf{E} = E_z \mathbf{i}_z$.



Then from $\nabla \cdot \mathbf{J} = \nabla \cdot \left[\dfrac{\sigma_0}{1 + z/d} E_z \mathbf{i}_z \right] = 0$,

E_z must be of the form $E_0 \left(1 + \frac{z}{d}\right)$. To find E_0, we write $-V_0 = \int_{z=0}^{d} E_0 \left(1 + \frac{z}{d}\right) \mathbf{i}_z \cdot dz\, \mathbf{i}_z$.

Thus we obtain

$\mathbf{E} = -\dfrac{2}{3} \dfrac{V_0}{d} \left(1 + \dfrac{z}{d}\right) \mathbf{i}_z$ which then gives

$\mathbf{J} = \sigma \mathbf{E} = -\dfrac{2}{3} \dfrac{\sigma_0 V_0}{d} \mathbf{i}_z$ and $\mathbf{D} = \epsilon \mathbf{E} = -\dfrac{8}{3} \dfrac{\epsilon_0 V_0}{d} \left(1 + \dfrac{z}{d}\right) \mathbf{i}_z$.

(b) $\rho_s = \mathbf{D} \cdot \mathbf{i}_n = -\dfrac{8}{3} \dfrac{\epsilon_0 V_0}{d}$ for $z = 0$ and $\dfrac{16}{3} \dfrac{\epsilon_0 V_0}{d}$ for $z = d$.

(c) $\rho = \nabla \cdot \mathbf{D} = -\dfrac{8}{3} \dfrac{\epsilon_0 V_0}{d^2}$.

(d) $\mathbf{P} = \mathbf{D} - \epsilon_0 \mathbf{E} = -2 \dfrac{\epsilon_0 V_0}{d} \left(1 + \dfrac{z}{d}\right) \mathbf{i}_z$

$\rho_{ps} = \mathbf{P} \cdot \mathbf{i}_n = \dfrac{2 \epsilon_0 V_0}{d}$ for $z = 0$ and $-\dfrac{4 \epsilon_0 V_0}{d}$ for $z = d$

$\rho_p = -\nabla \cdot \mathbf{P} = \dfrac{2 \epsilon_0 V_0}{d^2}$.

(e) From symmetry considerations, $\mathbf{H}_i = H_{\phi i} \mathbf{i}_\phi$. Applying Ampere's circuital law to a circular path of radius $r < a$ around the axis of the slab, we then have $2\pi r H_{\phi i} = -\frac{2}{3}\frac{\sigma_0 V_0}{d}(\pi r^2)$ or

$$\underset{\sim}{H}_i = -\frac{1}{3}\frac{\sigma_0 V_0 r}{d}\underset{\sim}{i}_\phi \quad \text{and} \quad \underset{\sim}{B}_i = \mu \underset{\sim}{H}_i = -\frac{2}{3}\frac{\mu_0 \sigma_0 V_0 r}{d}\underset{\sim}{i}_\phi .$$

(f) The current I drawn from the battery must be equal to the total current flowing inside the material from $z = d$ to $z = 0$. Thus

$$I = \frac{2}{3}\frac{\sigma_0 V_0}{d}\pi a^2, \text{ and } \underset{\sim}{H}_0 = -\frac{I}{2\pi r}\underset{\sim}{i}_\phi = -\frac{1}{3}\frac{\sigma_0 V_0 a^2}{rd}\underset{\sim}{i}_\phi .$$

(g) $$\underset{\sim}{J}_s = \underset{\sim}{i}_n \times (\underset{\sim}{H}_0 - \underset{\sim}{H}_i) = \begin{cases} -\frac{1}{3}\frac{\sigma_0 V_0}{d}\left(\frac{a^2}{r} - r\right)\underset{\sim}{i}_r & \text{for } z = 0 \\ \frac{1}{3}\frac{\sigma_0 V_0}{d}\left(\frac{a^2}{r} - r\right)\underset{\sim}{i}_r & \text{for } z = d \end{cases}$$

(h) $$\underset{\sim}{M} = \frac{\underset{\sim}{B}_i}{\mu_0} - \underset{\sim}{H}_i = \left(\frac{\mu}{\mu_0} - 1\right)\underset{\sim}{H}_i = -\frac{1}{3}\frac{\sigma_0 V_0 r}{d}\underset{\sim}{i}_\phi$$

$\underset{\sim}{J}_{ms} = \underset{\sim}{M} \times \underset{\sim}{i}_n = \frac{1}{3}\frac{\sigma_0 V_0 r}{d}\underset{\sim}{i}_r$ for $z = 0$, $-\frac{1}{3}\frac{\sigma_0 V_0 r}{d}\underset{\sim}{i}_r$ for $z = d$, and

$\frac{1}{3}\frac{\sigma_0 V_0 a}{d}\underset{\sim}{i}_z$ for $r = a$.

$$\underset{\sim}{J}_m = \underset{\sim}{\nabla} \times \underset{\sim}{M} = -\frac{2}{3}\frac{\sigma_0 V_0}{d}\underset{\sim}{i}_z .$$

(i) $$p_d = \underset{\sim}{J} \cdot \underset{\sim}{E} = \frac{4}{9}\frac{\sigma_0 V_0^2}{d^2}\left(1 + \frac{z}{d}\right)$$

$$P_d = \int p_d \, dv = \frac{2}{3}\frac{\pi a^2 \sigma_0 V_0^2}{d} \quad \text{and} \quad G = \frac{P_d}{V_0^2} = \frac{2}{3}\frac{\pi a^2 \sigma_0}{d} .$$

(j) $$w_e = \frac{1}{2}\underset{\sim}{D} \cdot \underset{\sim}{E} = \frac{8}{9}\epsilon_0 \frac{V_0^2}{d^2}\left(1 + \frac{z}{d}\right)^2$$

$$W_e = \int w_e \, dv = \frac{64}{27}\pi a^2 \frac{\epsilon_0 V_0^2}{d} \quad \text{and} \quad C = \frac{2W_e}{V_0^2} = \frac{128}{27}\frac{\pi a^2 \epsilon_0}{d} .$$

(k) $$w_{mi} = \frac{1}{2}\underset{\sim}{H}_i \cdot \underset{\sim}{B}_i = \frac{1}{9}\frac{\mu_0 \sigma_0^2 V_0^2 r^2}{d^2}$$

$$W_{mi} = \int w_{mi} \, dv = \frac{\mu_0 \sigma_0^2 V_0^2 \pi a^2}{18 d} \quad \text{and} \quad L_i = \frac{2W_{mi}}{I^2} = \frac{\mu_0 d}{4\pi} .$$

(l) $$[\underset{\sim}{P}]_{r=a} = [\underset{\sim}{E}]_{r=a} \times [\underset{\sim}{H}]_{r=a} = -\frac{2}{9}\frac{\sigma_0 V_0^2 a}{d^2}\left(1 + \frac{z}{d}\right)\underset{\sim}{i}_r$$

$[\underset{\sim}{P}]_{z=0}$ and $[\underset{\sim}{P}]_{z=d}$ are tangential to the surfaces and hence do not contribute to the power flow into the material. Thus

$$P_{in} = \int_{z=0}^{d}\int_{\phi=0}^{2\pi} [\underset{\sim}{P}]_{r=a} \cdot [-a \, d\phi \, dz \, \underset{\sim}{i}_r] = \frac{2}{3}\frac{\pi a^2 \sigma_0 V_0^2}{d} .$$

6.27. Effective area of air gap $= \pi(0.798 + 0.05)^2 = 2.259$ cm^2.

$B_g = \frac{3 \times 10^{-4}}{2.259 \times 10^{-4}}$ wb/m^2, $H_g = \frac{B_g}{\mu_0} = 0.1057 \times 10^7$ amp-turns/m.

B_a, flux density in the material $= \frac{3 \times 10^{-4}}{2 \times 10^{-4}} = 1.5$ wb/m^2.

From the B-H curve for annealed sheet steel, $H_a = 1000$ amp-turns/m.

Thus $NI = 1000 \times 20 \times 10^{-2} + 0.1057 \times 10^7 \times 0.1 \times 10^{-2} = 1257$ amp-turns.

6.28. From magnetic circuit considerations, $H_1 l_1 + H_2 l_2 = NI$ or $2H_1 + H_2 = 1500$.

From symmetry considerations, $\psi_2 = 2\psi_3$ or $B_2 A_2 = 2 B_3 A_3$ or $B_2 = 1.2 B_1$.

These two equations have to be solved for B_2. Since the B versus H relationship is available in the form of a graph, we have to use a trial and error method. This gives $B_2 \approx 1.47$ wb/m².

6.29. $P = \oint (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = \int_{y=0}^{w} \int_{x=0}^{d} [\mathbf{E} \times \mathbf{H}]_{z=0} \cdot dx\, dy\, \mathbf{i}_z - \int_{y=0}^{w} \int_{x=0}^{d} [\mathbf{E} \times \mathbf{H}]_{z=l} \cdot dx\, dy\, \mathbf{i}_z$

$$= [E_x]_{z=0} [H_y]_{z=0} (wd) - [E_x]_{z=l} [H_y]_{z=l} (wd)$$

But under the quasistatic approximation, $[E_x]_{z=0} \approx [E_x]_{z=l} = \frac{V(t)}{d}$.

Also, applying the integral form of Maxwell's equation for the curl of $\mathbf{H}$ to the rectangular path surrounding the crosssection of the structure in the $x =$ constant plane, we get

$$w\{[H_y]_{z=0} - [H_y]_{z=l}\} = \frac{d}{dt}[D_x (wl)]$$

or $\quad [H_y]_{z=0} - [H_y]_{z=l} = \frac{d}{dt}\left[\frac{\epsilon V}{d} l\right]$.

Thus $\quad P = \frac{V(t)}{d} \cdot \frac{d}{dt}\left[\frac{\epsilon V}{d} l\right] \cdot wd = V \frac{d}{dt}\left[\frac{\epsilon w l}{d} V\right] = \frac{d}{dt}\left(\frac{1}{2} C V^2\right)$.

6.30. $P = \oint (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = \int_{x=0}^{d} \int_{y=0}^{w} [\mathbf{E} \times \mathbf{H}]_{z=0} \cdot dx\, dy\, \mathbf{i}_z = [E_x]_{z=0} [H_y]_{z=0} (wd)$.

But under the quasistatic approximation, $[H_y]_{z=0} = \frac{I(t)}{w}$. Also, applying Faraday's law to the rectangular path surrounding the crosssection of the structure in the $y =$ constant plane, we get

$$[E_x]_{z=0}\, d = \frac{d}{dt}[B_y\, ld] \qquad \text{or} \qquad [E_x]_{z=0} = \frac{d}{dt}\left[\frac{\mu I l}{w}\right]$$

Thus $\quad P = \frac{I(t)}{w} \cdot \frac{d}{dt}\left[\frac{\mu I l}{w}\right] \cdot wd = I \frac{d}{dt}\left[\frac{\mu d l}{w} I\right] = \frac{d}{dt}\left(\frac{1}{2} L I^2\right)$.

6.31. Solution is completly analogous to that Example 6-16. See also page 537 of the text.

6.32. With reference to Figure 6.13(a) and the notation used in Example 6-16, we have $\bar{E}_x^q = \frac{\bar{V}_0}{d}$. Then applying $\nabla \times \bar{\mathbf{H}} = \bar{\mathbf{J}} = \sigma \bar{\mathbf{E}}$ and $\nabla \times \bar{\mathbf{E}} = -j\omega \bar{\mathbf{B}}$ successively, and evaluating the arbitrary constants of integration from the conditions that $[\bar{H}_y']_{z=l} = [\bar{H}_y'']_{z=l} = \cdots = 0$ since the current flowing on the perfect conductor surfaces at $z = l$ must be zero, and

$[\bar{E}_x']_{z=0} = [\bar{E}_x'']_{z=0} = \cdots = 0$ since the condition that the voltage at $z=0$ must be equal to the source voltage is satisfied by $\bar{E}_x^q$ alone, we get

$$\bar{H}_y' = -\frac{\sigma \bar{V}_0}{d}(z-l)$$

$$\bar{E}_x' = j\omega\mu\sigma \frac{\bar{V}_0}{d}\left[\frac{(z-l)^2}{2} - \frac{l^2}{2}\right]$$

$$\bar{H}_y'' = -j\omega\mu\sigma^2 \frac{\bar{V}_0}{d}\left[\frac{(z-l)^3}{6} - \frac{l^2}{2}(z-l)\right]$$

$$\bar{E}_x'' = -\omega^2\mu^2\sigma^2 \frac{\bar{V}_0}{d}\left[\frac{(z-l)^4}{24} - \frac{l^2(z-l)^2}{4} + \frac{5l^4}{24}\right]$$

$$\bar{H}_y''' = \omega^2\mu^2\sigma^3 \frac{\bar{V}_0}{d}\left[\frac{(z-l)^5}{120} - \frac{l^2(z-l)^3}{12} + \frac{5l^4(z-l)}{24}\right]$$

and so on. The total magnetic field at $z=0$ is then given by

$$[\bar{H}_y]_{z=0} = [\bar{H}_y']_{z=0} + [\bar{H}_y'']_{z=0} + [\bar{H}_y''']_{z=0} + \cdots$$

$$= \frac{\bar{V}_0}{d}\frac{(1-j)}{\sqrt{2}}\sqrt{\frac{\sigma}{\omega\mu}}\tanh\left[(1+j)\sqrt{\frac{\omega\mu\sigma}{2}}\,l\right]$$

The phasor current drawn from the voltage source is

$$\bar{I}_0 = [\bar{I}]_{z=0} = [\bar{H}_y]_{z=0}\,w = \frac{\bar{V}_0 w}{d}\frac{1-j}{\sqrt{2}}\sqrt{\frac{\sigma}{\omega\mu}}\tanh\left[(1+j)\sqrt{\frac{\omega\mu\sigma}{2}}\,l\right].$$

Now, for $\sqrt{\frac{\omega\mu\sigma}{2}}\,l \ll 1$, $\tanh\left[(1+j)\sqrt{\frac{\omega\mu\sigma}{2}}\,l\right] \approx (1+j)\sqrt{\frac{\omega\mu\sigma}{2}}\,l$ and

$\bar{I}_0 \approx \bar{V}_0 \frac{\sigma w l}{d} = \bar{V}_0 G$. Thus the structure behaves at its input as a single resistor for the condition $\sqrt{\frac{\omega\mu\sigma}{2}}\,l \ll 1$. For copper $\sigma = 5.8\times10^7$ mhos/m, this condition reduces to $f \ll 0.00437/l^2$. For $l = 1$ cm, $f \ll 43.7$ Hz.

To examine the input behavior of the structure for frequencies slightly beyond the quasistatic approximation, we consider one more term in the expansion for $\tanh\left[(1+j)\sqrt{\frac{\omega\mu\sigma}{2}}\,l\right]$ and find that the input behavior is equivalent to the series combination of a resistor $\frac{1}{G}$ and an inductor $\frac{1}{3}L$ where $L = \frac{\mu d l}{w}$. For frequencies for which $f \gg \frac{1}{\pi\mu\sigma l^2}$,

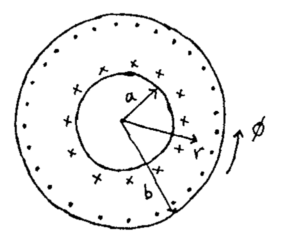
$\tanh\left[(1+j)\sqrt{\frac{\omega\mu\sigma}{2}}\,l\right] \approx 1$ and the input behavior is equivalent to the series combination of a resistor of value $\sqrt{\frac{\pi f\mu}{\sigma}}\,\frac{d}{w}$ and an inductor of value $\sqrt{\frac{\mu}{4\pi f\sigma}}\,\frac{d}{w}$.

6.33. $\frac{1}{2\pi l\sqrt{\mu\epsilon}} = \frac{1}{2\pi(1)\sqrt{\mu_0\epsilon_0}} = 0.4775\times10^8$ Hz

(a) $f = 150$ Hz $\ll \frac{1}{2\pi l\sqrt{\mu\epsilon}}$. Hence, the structure behaves like a single inductor of value $L = \frac{\mu\, d\, l}{w} = \frac{\mu_0 (0.2)(1)}{(0.5)} = 1.6\pi \times 10^{-7}$ h as viewed by the current source. The voltage developed across the current source is $L\frac{dI}{dt} = -480\pi^2 \times 10^{-7} \sin 300\pi t$.

(b) $f = 150$ MHz is comparable to $\frac{1}{2\pi l\sqrt{\mu\epsilon}}$. Hence, we have to use Eq. (6-151). Thus, since $\omega\sqrt{LC} = \omega\sqrt{\mu\epsilon}\, l = \omega\sqrt{\mu\epsilon} = \pi$, $\bar{V} = 0$.

6.34. Recognizing that $\underset{\sim}{E} = E_r(r, z, t)\, \underset{\sim}{i}_r$ and $\underset{\sim}{H} = H_\phi(r, z, t)\, \underset{\sim}{i}_\phi$ and substituting in the two Maxwell's curl equations, we get



$$\frac{\partial E_r}{\partial z} = -\mu \frac{\partial H_\phi}{\partial t} \quad \text{and} \quad \frac{1}{r}\frac{\partial}{\partial z}(r H_\phi) = \sigma E_r + \epsilon \frac{\partial E_r}{\partial t}.$$

But from $V(z,t) = \int_{r=a}^{b} E_r(r,z,t)$ and since $E_r \propto \frac{1}{r}$ from Gauss' law, we have $E_r(r,z,t) = \frac{V(z,t)}{r \ln\frac{b}{a}}$. Also, from $I(z,t) = \int_{\phi=0}^{2\pi} H_\phi(r,z,t)\, r\, d\phi$, we have $H_\phi(r,z,t) = \frac{I(z,t)}{2\pi r}$. Substituting these into the differential equations and simplifying, we get

$$\frac{\partial V}{\partial z} = -\frac{\partial}{\partial t}\left[\left(\frac{\mu}{2\pi}\ln\frac{b}{a}\right) I\right] = -\frac{\partial}{\partial t}(\mathcal{L} I).$$

$$\frac{\partial I}{\partial z} = -\left(\frac{2\pi\sigma}{\ln\frac{b}{a}}\right) V - \frac{\partial}{\partial t}\left(\frac{2\pi\epsilon}{\ln\frac{b}{a}} V\right) = -\mathcal{G} V - \frac{\partial}{\partial t}(\mathcal{C} V).$$

$$P(z,t) = \int_{r=a}^{b}\int_{\phi=0}^{2\pi} \frac{V(z,t)}{r\ln\frac{b}{a}}\, \underset{\sim}{i}_r \times \frac{I(z,t)}{2\pi r}\, \underset{\sim}{i}_\phi \cdot \underset{\sim}{i}_z\, r\, dr\, d\phi = V(z,t)\cdot I(z,t).$$

6.35. Circuit (a) follows from

$$\lim_{\Delta z \to 0} \frac{V(z+\frac{\Delta z}{2}, t) - V(z-\frac{\Delta z}{2}, t)}{\Delta z} = -\lim_{\Delta z\to 0}\left[\frac{1}{2}\mathcal{L}\frac{\partial I(z-\frac{\Delta z}{2},t)}{\partial t} + \frac{1}{2}\mathcal{L}\frac{\partial I(z+\frac{\Delta z}{2},t)}{\partial t}\right].$$

$$\lim_{\Delta z\to 0}\frac{I(z+\frac{\Delta z}{2},t) - I(z-\frac{\Delta z}{2},t)}{\Delta z} = -\mathcal{G}V(z,t) - \mathcal{C}\frac{\partial V(z,t)}{\partial t}.$$

Circuit (b) follows from

$$\lim_{\Delta z\to 0}\frac{V(z+\frac{\Delta z}{2},t) - V(z-\frac{\Delta z}{2},t)}{\Delta z} = -\mathcal{L}\frac{\partial I(z,t)}{\partial t}$$

$$\lim_{\Delta z\to 0}\frac{I(z+\frac{\Delta z}{2},t) - I(z,t)}{\Delta z} = \lim_{\Delta z\to 0}\left[-\frac{\mathcal{G}}{2}V(z+\frac{\Delta z}{2},t) - \frac{\mathcal{C}}{2}\frac{\partial V(z+\frac{\Delta z}{2},t)}{\partial t}\right]$$

$$\lim_{\Delta z\to 0}\frac{I(z,t) - I(z-\frac{\Delta z}{2},t)}{\Delta z} = \lim_{\Delta z\to 0}\left[-\frac{\mathcal{G}}{2}V(z-\frac{\Delta z}{2},t) - \frac{\mathcal{C}}{2}\frac{\partial V(z-\frac{\Delta z}{2},t)}{\partial t}\right]$$

6.36. $\nabla \times \nabla \times \underset{\sim}{H} = \nabla \times \dfrac{\partial \underset{\sim}{D}}{\partial t}$

$$\nabla(\nabla \cdot \underset{\sim}{H}) - \nabla^2 \underset{\sim}{H} = \frac{\partial}{\partial t}(\nabla \times \underset{\sim}{D}) = -\mu\epsilon \frac{\partial}{\partial t}\left(\frac{\partial \underset{\sim}{H}}{\partial t}\right)$$

since $\nabla \cdot \underset{\sim}{H} = \frac{1}{\mu} \nabla \cdot \underset{\sim}{B} = 0$, it follows that $\nabla^2 \underset{\sim}{H} = \mu\epsilon \dfrac{\partial^2 \underset{\sim}{H}}{\partial t^2}$.

6.37. Assuming $E_x(z,t) = Z(z) \cdot T(t)$ and substituting in the differential equation and separating variables, we get $Z'' = \mu\epsilon\alpha^2 Z$ and $T'' = \alpha^2 T$. Thus

$Z = A e^{\alpha\sqrt{\mu\epsilon}\, z} + B e^{-\alpha\sqrt{\mu\epsilon}\, z}$ and $T = C e^{\alpha t} + D e^{-\alpha t}$ and

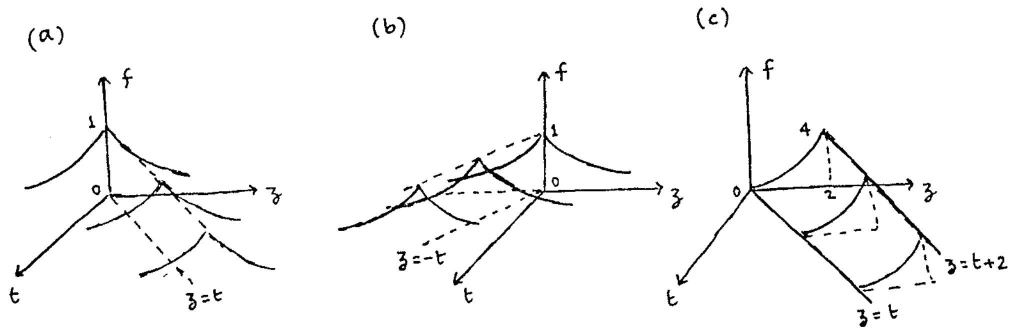
$$E_x(z,t) = Z\,T = A' e^{\alpha(t+\sqrt{\mu\epsilon}\, z)} + B' e^{\alpha(t-\sqrt{\mu\epsilon}\, z)} + C' e^{-\alpha(t+\sqrt{\mu\epsilon}\, z)} + D' e^{-\alpha(t-\sqrt{\mu\epsilon}\, z)}$$

Since α can assume several values, this solution shows that $E_x(z,t)$ can be a superposition of arbitrary functions of $(t+\sqrt{\mu\epsilon}\, z)$ and $(t-\sqrt{\mu\epsilon}\, z)$. Alternatively, by defining $\tau = z\sqrt{\mu\epsilon}$, we can write the differential equation as

$$\frac{\partial^2 E_x}{\partial \tau^2} - \frac{\partial^2 E_x}{\partial t^2} = 0 \quad \text{or} \quad \left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial t}\right) E_x = 0 \quad \text{which gives}$$

$$\frac{\partial E_x}{\partial \tau} = \pm \frac{\partial E_x}{\partial t} \quad \text{or } E_x(z,t) = A f(t-\tau) + B g(t-\tau) = A f(t-\sqrt{\mu\epsilon}\, z) + B g(t+\sqrt{\mu\epsilon}\, z).$$

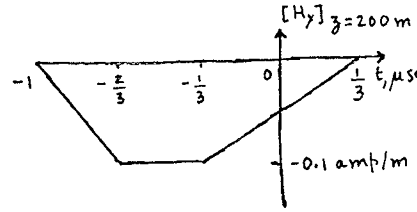
6.38.



6.39. (a) By application of Gauss' law to a spherical surface concentric with the balloon, we get a value of zero for the field at points inside the balloon and $\dfrac{Q}{4\pi\epsilon_0 r^2}\, \underset{\sim}{i}_r$ for points outside the balloon. Thus the values of $\underset{\sim}{E}$ in the different regions are as given on page 538 of the text.

(b) Since $\underset{\sim}{E}$ does not vary with time, there is no wave propagation.

6.40. Since the velocity of propagation is 3×10^8 m/sec, a field intensity which exists at a value of z at $t=0$ must exist at $z = 200$ m at $t = (z-200)/300$ μsec. Also, the magnetic field is in the y direction and its value is $-\frac{E_x}{377}$. Thus we get the required time variation of $[H_y]_{z=200m}$ as shown in the figure.



6.41. See page 538 of the text for answers.

6.42. (a) Since $\underset{\sim}{\beta} = 0.04\pi(\sqrt{3}\,\underset{\sim}{i}_x - 2\underset{\sim}{i}_y - 3\underset{\sim}{i}_z)$, the direction of propagation is along the unit vector $\frac{1}{4}(\sqrt{3}\,\underset{\sim}{i}_x - 2\underset{\sim}{i}_y - 3\underset{\sim}{i}_z)$.

(b) $\beta = 0.04\pi\,|\sqrt{3}\,\underset{\sim}{i}_x - 2\underset{\sim}{i}_y - 3\underset{\sim}{i}_z| = 0.16\pi$

$\lambda = 2\pi/\beta = 12.5$ m

(c) Since the medium is free space, $v_p = 3\times 10^8$ m/sec, and

$f = v_p/\lambda = 24$ MHz.

(d) $\lambda_x = \frac{2\pi}{\beta_x} = 28.87$ m, $\lambda_y = \frac{2\pi}{\beta_y} = 25$ m, $\lambda_z = \frac{2\pi}{\beta_z} = 16\frac{2}{3}$ m

$v_{px} = \frac{\omega}{\beta_x} = 6.928\times10^8$ m/sec, $v_{py} = \frac{\omega}{\beta_y} = 6\times10^8$ m/sec, $v_{pz} = \frac{\omega}{\beta_z} = 4\times10^8$ m/sec.

(e) Linearly polarized along the direction of $(-\underset{\sim}{i}_x - 2\sqrt{3}\,\underset{\sim}{i}_y + \sqrt{3}\,\underset{\sim}{i}_z)$.

(f) $\bar{\underset{\sim}{H}} = \frac{1}{\omega\mu}\underset{\sim}{\beta}\times\underset{\sim}{\bar{E}} = -\frac{1}{60\pi}(\sqrt{3}\,\underset{\sim}{i}_x + \underset{\sim}{i}_z)\,e^{-j0.04\pi(\sqrt{3}x - 2y - 3z)}$.

6.43. (a) Since $\underset{\sim}{\beta}\cdot\underset{\sim}{E}_0 = 0$ for the given vector, it represents the electric field vector of a uniform plane wave.

(b) See page 538 of the text for answers. To determine the sense of polarization, we note that

$$\underset{\sim}{E}_0 = \mathcal{R}e\left[(-\sqrt{3}\,\underset{\sim}{i}_x + \underset{\sim}{i}_y)e^{j\omega t} + (-\tfrac{1}{2}\underset{\sim}{i}_x - \tfrac{\sqrt{3}}{2}\underset{\sim}{i}_y + \sqrt{3}\,\underset{\sim}{i}_z)e^{j\frac{\pi}{2}}e^{j\omega t}\right]$$

$$= (-\sqrt{3}\,\underset{\sim}{i}_x + \underset{\sim}{i}_y)\cos\omega t + (\tfrac{1}{2}\underset{\sim}{i}_x + \tfrac{\sqrt{3}}{2}\underset{\sim}{i}_y - \sqrt{3}\,\underset{\sim}{i}_z)\sin\omega t.$$

By noting the orientations of $\underset{\sim}{E}_0$ for $\omega t = 0$ and $\omega t = \frac{\pi}{2}$ relative to the direction of propagation, we find that the polarization is left circular.

6.44. (a) For the given $\bar{\underset{\sim}{E}}$ and $\bar{\underset{\sim}{H}}$, $\bar{\underset{\sim}{E}}_0\cdot\bar{\underset{\sim}{H}}_0 = 0$, $\underset{\sim}{\beta}\cdot\bar{\underset{\sim}{E}}_0 = 0$, and $\underset{\sim}{\beta}\cdot\bar{\underset{\sim}{H}}_0 = 0$. Hence, the given field vectors represent a uniform plane wave.

(b) Direction of propagation is along the unit vector $\frac{\sqrt{3}}{2}\underset{\sim}{i}_x + \frac{1}{2}\underset{\sim}{i}_z$.

From $\beta = 0.05\pi\,|\sqrt{3}\,\underset{\sim}{i}_x + \frac{1}{2}\underset{\sim}{i}_z| = 0.1\pi$, we obtain $\lambda = \frac{2\pi}{\beta} = 20\text{m}$.

To find the velocity of propagation, we first find the frequency by using $\bar{\underset{\sim}{H}}_0 = \frac{1}{\omega\mu}\underset{\sim}{\beta} \times \bar{\underset{\sim}{E}}_0$. Thus

$$\underset{\sim}{\beta} \times \bar{\underset{\sim}{E}}_0 = 0.1\pi\,(\underset{\sim}{i}_x - j2\,\underset{\sim}{i}_y - \sqrt{3}\,\underset{\sim}{i}_z)$$

$$f = \frac{1}{2\pi\mu_0}\frac{\underset{\sim}{\beta} \times \bar{\underset{\sim}{E}}_0}{\bar{\underset{\sim}{H}}_0} = 7.5\text{ MHz}.$$

Then $v = \lambda f = 1.5 \times 10^8$ m/sec.

By noting that $\bar{\underset{\sim}{E}}_0 = -j\,(\underset{\sim}{i}_x - \sqrt{3}\,\underset{\sim}{i}_z) - 2\,\underset{\sim}{i}_y$, we find that the wave is circularly polarized. Furthermore,

$\underset{\sim}{E}_0 = (\underset{\sim}{i}_x - \sqrt{3}\,\underset{\sim}{i}_z)\sin\omega t - 2\cos\omega t\,\underset{\sim}{i}_y$ so that

$[\underset{\sim}{E}_0]_{\omega t=0} = -2\,\underset{\sim}{i}_y$ and $[\underset{\sim}{E}_0]_{\omega t=\frac{\pi}{2}} = \underset{\sim}{i}_x - \sqrt{3}\,\underset{\sim}{i}_z$. Since the direction of propagation is along $\frac{\sqrt{3}}{2}\underset{\sim}{i}_x + \frac{1}{2}\underset{\sim}{i}_z$, the polarization of the wave is then right circular.

6.45. $\frac{1}{\sqrt{\mathcal{L}\mathcal{C}}} = \frac{L}{\sqrt{LC}} = \frac{L\omega}{\sqrt{(\omega L)(\omega C)}}$ has the units of meters/sec.

$\sqrt{\mathcal{L}/\mathcal{C}} = \sqrt{L/C} = \sqrt{(\omega L)/(\omega C)}$ has the units of ohms.

6.46. The reflection coefficient at the perfect conductor surface is -1 for E_x and $+1$ for H_y. The velocity of propagation is 3×10^8 m/sec or 300 m/μsec. The intrinsic impedance of the medium is 377 ohms. Using this information and sketching the (+) and (−) wave fields versus z and adding them up, we get the total fields versus z for the specified times as follows:

t, μsec	$\frac{150}{37.7}E_x$, volts/m	$1500\,H_y$, amps/m	z, m
1/4	$z + 375$	$z + 375$	$-375 < z < -225$
	$-(z + 75)$	$-(z + 75)$	$-225 < z < -75$
3/4	$z + 225$	$z + 225$	$-225 < z < -75$
	$-2z$	150	$-75 < z < 0$
1/2	0	$2(z + 150)$	$-150 < z < 0$

$t, \mu sec$	$\frac{150}{37.7} E_x$, volts/m	$1500\, H_y$, amps/m	z, m
$1\frac{1}{4}$	$-(z+225)$	$(z+225)$	$-225 < z < -75$
	$2z$	150	$-75 < z < 0$
2	$-(z+450)$	$(z+450)$	$-450 < z < -300$
	$(z+150)$	$-(z+150)$	$-300 < z < -150$

Similarly, E_x and H_y versus t in the plane $z = -150$ m are given by

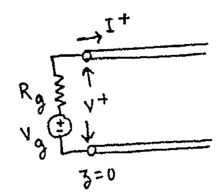
$$[E_x]_{z=-150} = \begin{cases} 75.4 & 0 < t < 0.5 \\ 75.4\,(1-t) & 0.5 < t < 1.5 \\ 75.4\,(t-2) & 1.5 < t < 2 \end{cases}$$

$$[H_y]_{z=-150} = \begin{cases} 0.2\,t & 0 < t < 0.5 \\ 0.2\,(1-t) & 0.5 < t < 1 \\ 0.2\,(t-1) & 1 < t < 1.5 \\ 0.2\,(2-t) & 1.5 < t < 2 \end{cases}$$

where E_x is in volts/m, H_y is in amps/m, t is in μsec, and z is in m.

6.47. By arguments similar to those in Example 6-20 using the bounce diagram technique, we obtain the answers given on page 538 of the text.

6.48. (a) $V_g - R_g I^+ = V^+$, $\quad I^+ = \dfrac{V^+}{Z_0}$

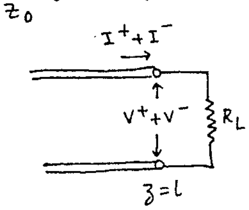


solving these two equations, we get $V^+ = \dfrac{Z_0}{R_g + Z_0} V_g$

which gives for $t > 0$,

$$V^+(z,t) = \frac{Z_0}{R_g+Z_0} V_g\left(t - \frac{z}{v}\right) \text{ and } I^+(z,t) = \frac{1}{R_g+Z_0} V_g\left(t - \frac{z}{v}\right)$$

(b) $V^+ + V^- = R_L\,(I^+ + I^-)$



$I^+ = \dfrac{V^+}{Z_0}$, $\quad I^- = -\dfrac{V^-}{Z_0}$

Solving these three equations, we get $V^- = \dfrac{R_L - Z_0}{R_L + Z_0} V^+$

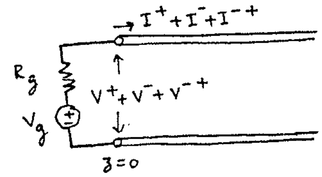
which gives for $t > l/v$,

$$V^-(z,t) = \frac{Z_0}{R_g+Z_0}\,\Gamma_R\, V_g\left(t - \frac{l}{v} - \frac{l-z}{v}\right) = \frac{Z_0}{R_g+Z_0}\,\Gamma_R\, V_g\left(t - \frac{2l}{v} + \frac{z}{v}\right)$$

$$I^-(z,t) = -\frac{1}{R_g+Z_0}\,\Gamma_R\, V_g\left(t - \frac{2l}{v} + \frac{z}{v}\right).$$

(c) $V_g - R_g(I^+ + I^- + I^{-+}) = V^+ + V^- + V^{-+}$

$I^+ = \dfrac{V^+}{Z_0}, \quad I^- = -\dfrac{V^-}{Z_0}, \quad I^{-+} = \dfrac{V^{-+}}{Z_0}$



Solving these four equations using the result for V^+ from (a), we get

$V^{-+} = V^- \dfrac{R_g - Z_0}{R_g + Z_0}$ which gives for $t > \frac{2l}{v}$,

$$V^{-+}(z,t) = \frac{Z_0}{R_g + Z_0}\,\Gamma_R \Gamma_g\, V_g\left(t - \frac{2l}{v} - \frac{z}{v}\right)$$

$$I^{-+}(z,t) = \frac{1}{R_g + Z_0}\,\Gamma_R \Gamma_g\, V_g\left(t - \frac{2l}{v} - \frac{z}{v}\right)$$

(d) Writing expressions for $V^{-+-}(z,t), \cdots$ and $I^{-+-}(z,t), \cdots$, and noting that the steady state voltage and current are the superpositions of the voltages and currents, respectively, associated with the transient waves, we obtain the series expressions for $V_{ss}(z,t)$ and $I_{ss}(z,t)$ given in the problem.

(e) (i) $$V_{ss}(z,t) = \frac{Z_0 V_0}{R_g + Z_0}\left[\sum_{n=0}^{\infty}(\Gamma_R\Gamma_g)^n + \Gamma_R \sum_{n=0}^{\infty}(\Gamma_R\Gamma_g)^n\right]$$

$$= \frac{V_0 Z_0}{R_g + Z_0}\,\frac{1 + \Gamma_R}{1 - \Gamma_R\Gamma_g} = V_0 \frac{R_L}{R_L + R_g}$$

$$I_{ss}(z,t) = \frac{V_0}{R_g + Z_0}\left[\sum_{n=0}^{\infty}(\Gamma_R\Gamma_g)^n - \Gamma_R \sum_{n=0}^{\infty}(\Gamma_R\Gamma_g)^n\right]$$

$$= \frac{V_0}{R_g + Z_0}\,\frac{1 - \Gamma_R}{1 - \Gamma_R\Gamma_g} = \frac{V_0}{R_L + R_g}.$$

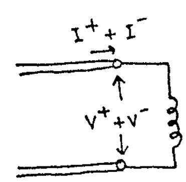
(ii) $$V_{ss}(z,t) = V_0 \frac{Z_0}{R_g + Z_0}\,\frac{\cos[\omega(t - z/v) - \alpha] + \Gamma_R \cos[\omega(t + \frac{z}{v} - \frac{2l}{v}) - \alpha]}{[1 + (\Gamma_R\Gamma_g)^2 - 2\Gamma_R\Gamma_g \cos\omega\frac{2l}{v}]^2}$$

$$I_{ss}(z,t) = \frac{V_0}{R_g + Z_0}\,\frac{\cos[\omega(t - z/v) - \alpha] - \Gamma_R \cos[\omega(t + \frac{z}{v} - \frac{2l}{v}) - \alpha]}{[1 + (\Gamma_R\Gamma_g)^2 - 2\Gamma_R\Gamma_g \cos\omega\frac{2l}{v}]^2}$$

where $\alpha = \tan^{-1} \dfrac{\Gamma_R\Gamma_g \sin\omega\frac{2l}{v}}{1 - \Gamma_R\Gamma_g \cos\omega\frac{2l}{v}}$.

6.49. Writing $V^+ + V^- = L\dfrac{d}{dt}(I^+ + I^-)$,

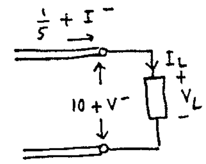
$V^+ = V_0$, $I^+ = \dfrac{V^+}{Z_0}$, and $I^- = -\dfrac{V^-}{Z_0}$, we obtain



$\dfrac{L}{Z_0}\dfrac{dV^-}{dt} + V^- = -V_0$ or $V^- = -V_0 + Ae^{-\frac{Z_0}{L}t}$

But $[I^+ + I^-]_{t=0+} = \left[\dfrac{V^+}{Z_0} - \dfrac{V^-}{Z_0}\right]_{t=0+} = 0$ or $[V^-]_{t=0+} = [V^+]_{t=0+} = V_0$.

Thus we obtain $A = 2V_0$ and $V^- = -V_0 + 2V_0 e^{-\frac{Z}{L}t}$.



6.50. $V_L = 10 + V^-$, $I_L = \frac{1}{5} + I^-$.

Thus $10 + V^- = 50\left(\frac{1}{5} + I^-\right)^2$

Substituting $I^- = -\frac{V^-}{Z_0} = -\frac{V^-}{50}$ and solving for V^-,

we obtain $V^- = -5.3$ or 75.3 volts. The second answer is ruled out since the reflected power cannot be greater than the incident power in view of the passive nature of the nonlinear element. Thus $V^- = -5.3$ volts.

6.51. See page 538 of the text for answers.

6.52. For line short-circuited at one end and open-circuited at the other end, $f_n = (2n+1)v_p/4l$, $n = 1, 2, 3, \ldots, \infty$, where v_p is the phase velocity. Voltage standing wave patterns are half sinusoids with zeros at the short-circuited end and maxima at the open-circuited end. Current standing wave patterns are half sinusoids with maxima at the short-circuited end and zeros at the open-circuited end.

For line open-circuited at both ends, $f_n = \frac{n v_p}{4l}$, $n = 1, 2, 3, \ldots, \infty$. Voltage standing wave patterns are half sinusoids with maxima at both ends and current standing wave patterns are half sinusoids with zeros at both ends.

6.53. (a) $l = \frac{\lambda}{4}$ for 500 Hz. Hence, $\bar{Z}_{in} = jZ_0 \tan\frac{2\pi}{\lambda}\cdot\frac{\lambda}{4} = \infty$. The current drawn from the source is zero. Voltage at source end of the line is equal to the source voltage. The standing wave pattern is therefore given by $|\bar{V}(d)| = 10 \sin\frac{\pi d}{2l}$. Knowing that $I_{max} = \frac{V_{max}}{Z_0}$, we then obtain the current standing wave pattern to be

$$|\bar{I}(d)| = \frac{1}{5}\cos\frac{\pi d}{2l}.$$

(b) $l = \frac{\lambda}{2}$ for 1000 Hz. Hence, $\bar{Z}_{in} = 0$. The current drawn from the source is simply the source voltage divided by the internal resistance. Thus the maximum current in the current standing wave pattern which occurs at either end of the line is $\frac{5}{100} = \frac{1}{20}$.

This gives $|\bar{I}(d)| = \frac{1}{20}\left|\cos\frac{\pi d}{l}\right|$. Now, knowing that $V_{max} = I_{max} Z_0$, we obtain $|\bar{V}(d)| = 2.5 \sin \frac{\pi d}{l}$.

(c) Substituting the given values for d in the expressions for $|\bar{V}(d)|$ and $|\bar{I}(d)|$ obtained in parts (a) and (b) and noting that the rms value of the superposition of two voltages of different but harmonically related frequencies is equal to the square root of the sum of the squares of the rms values of the individual components, we obtain the answers given on page 539 of the text.

6.54. Equating $j\omega \mathcal{L} l$ to $Z_{in} = jZ_0 \tan\beta l = j\sqrt{\frac{\mathcal{L}}{\mathcal{C}}}\tan\omega\sqrt{\mathcal{L}\mathcal{C}}\, l$, we have $\tan\beta l = \beta l$. The first two values of βl for which this equation is satisfied are approximately 4.49 and 7.72. This gives $f = 0.715\frac{v_p}{l}$ and $1.229\frac{v_p}{l}$ where v_p is the phase velocity.

6.55. $|\bar{V}(d)| = |\bar{V}^+|\,|1+\bar{\Gamma}(0)e^{-j2\beta d}| = |\bar{V}^+|\left[1+|\bar{\Gamma}(0)|^2+2|\bar{\Gamma}(0)|\cos(\theta-2\beta d)\right]^{1/2}$

where $\theta = \underline{/\bar{\Gamma}(0)}$. The derivative of $|\bar{V}(d)|$ with respect to d is

$$|\bar{V}^+|\frac{|\bar{\Gamma}(0)|\,2\beta\sin(\theta-2\beta d)}{\left[1+|\bar{\Gamma}(0)|^2+2|\bar{\Gamma}(0)|\cos(\theta-2\beta d)\right]^{1/2}}.$$

This quantity varies faster near the minima of the standing wave pattern than near the maxima of the pattern. This is because although $|\sin(\theta-2\beta d)|$ varies in the same manner near the minima and maxima, the quantity $\left[1+|\bar{\Gamma}(0)|^2+2|\bar{\Gamma}(0)|\cos(\theta-2\beta d)\right]^{1/2}$ has a smaller value near the minima than near the maxima. Hence, the minima of the standing wave pattern are sharper than the maxima.

6.56. At a voltage maximum, $\bar{V}$ and $\bar{I}$ are in phase and their magnitudes are $|\bar{V}| = |\bar{V}^+|\left[1+|\bar{\Gamma}(0)|\right]$ and $|\bar{I}| = \frac{|\bar{V}^+|}{Z_0}\left[1-|\bar{\Gamma}(0)|\right]$. Hence

$$Z = Z_0\frac{1+|\bar{\Gamma}(0)|}{1-|\bar{\Gamma}(0)|} = Z_0\,(\text{VSWR}).$$

At a voltage minimum, $\bar{V}$ and $\bar{I}$ are in phase and their magnitudes are $|\bar{V}| = |\bar{V}^+|\left[1-|\bar{\Gamma}(0)|\right]$ and $|\bar{I}| = \frac{|\bar{V}^+|}{Z_0}\left[1+|\bar{\Gamma}(0)|\right]$. Hence

$$\bar{Z} = Z_0\frac{1-|\bar{\Gamma}(0)|}{1+|\bar{\Gamma}(0)|} = Z_0/(\text{VSWR}).$$

6.57. Method is the same as for Example 6-23. See page 539 of the text for answers. Fraction of incident power transmitted into medium 3 is equal to $(1-|\bar{\Gamma}_3|^2) = 3/4$.

6.58. No standing waves in medium 3. From $\bar{\Gamma}_1 = -\frac{1}{5}$, VSWR in line 2 is 1.5. Since the electrical length of medium 2 is $\frac{3\lambda}{8}$, the standing wave pattern for $|\bar{E}_x|$ in that medium consists of of a minimum (say, 1) at the right end of the medium, a maximum of 1.5 at 2.5 cm from that end and reaches a value of $\frac{|1+\bar{\Gamma}_1 e^{-j\frac{4\pi}{\lambda}\cdot\frac{3\lambda}{8}}|}{(1-|\bar{\Gamma}_1|)} = 1.275$ at the left end of medium 2. $\bar{\Gamma}_2 = \bar{\Gamma}_1 e^{-j\frac{4\pi}{\lambda}\cdot\frac{3\lambda}{8}} = -j\frac{1}{5}$. Computing the line impedance at the left end of line 2 and using it to compute $\bar{\Gamma}_3$, we obtain $\bar{\Gamma}_3 = 0.3879 \underline{/207^\circ 9'}$ which gives VSWR in medium 1 to be 2.2675. Since the phase angle of $\bar{\Gamma}_3$ is $207^\circ 9'$, $|\bar{E}_x|$ is neither a minimum nor a maximum at the right end of medium 1. The first minimum of the standing wave pattern for $|\bar{E}_x|$ occurs at a distance d to the left of the interface given by $2\beta_1 d = 207^\circ 9' - 180^\circ = 27^\circ 9'$ or $d = 0.7542$ cm, and its value is $1.275\,\frac{1-|\bar{\Gamma}_3|}{1+|\bar{\Gamma}_3|} = 1.15$.

Proceeding further, we obtain a value of $1.15 \times 2.2675 = 2.6$ for the maximum of $|\bar{E}_x|$ located 5 cm to the left of the minimum. The corresponding standing wave patterns for $|\bar{H}_y|$ are as follows:

In medium 2, maximum of $3/\eta_0$ at the right end, minimum of $2/\eta_0$ at 2.5 cm from that end and reaching a value of $2.55/\eta_0$ at the left end; in medium 3, a maximum of $2.6/\eta_0$ at 0.7542 cm from the right end and a minimum of $1.15/\eta_0$ at 5 cm from the maximum, and so on.

Fraction of incident power transmitted into medium 3 $= 1-|\bar{\Gamma}_3|^2 = 1-0.3879^2 = 0.8454$.

Wave impedance in medium 1 at a distance of 4 cm or $\frac{4}{20}\lambda_1 \left(= \frac{\lambda_1}{5}\right)$ from the interface between media 1 and 2 is equal to

$$\eta_0 \frac{1+\bar{\Gamma}_3 e^{-j2\beta_1 \frac{\lambda_1}{5}}}{1-\bar{\Gamma}_3 e^{-j2\beta_1 \frac{\lambda_1}{5}}} = \eta_0 (1.062 + j\,0.8654).$$

6.59. From $\eta_2 = \sqrt{\eta_1 \eta_3}$, we obtain the permittivity of the quarter wave dielectric coating to be $4\epsilon_0$. Now, since the wavelength corresponding to 1500 MHz in this coating medium is 10 cm, its thickness must be 2.5 cm.

6.60. (a) From VSWR = 3.0, we get $|\bar{\Gamma}_R| = \frac{1}{2}$. From the given data, the distance between load and first voltage minimum is $\frac{15}{40}\lambda = \frac{3\lambda}{8}$ so that $\underline{/\bar{\Gamma}_R} = \frac{\pi}{2}$. Thus $\bar{\Gamma}_R = j\frac{1}{2}$ and $\bar{Z}_R = Z_0 \frac{1+\bar{\Gamma}_R}{1-\bar{\Gamma}_R} = (30 + j40)\,\Omega$.

(b) The quarter wave section must be placed at a voltage minimum or at a voltage maximum of the standing wave pattern. In this case, the first voltage minimum is at 15 cm from the load and since $\lambda = 20$ cm, there is a voltage maximum at 5 cm from the load. Hence, the quarter wave section must be placed at 5 cm from the load. Since the line impedance at this location is $Z_0(\text{VSWR}) = 150\,\Omega$, the characteristic impedance of the quarter wave section must be $\sqrt{50 \times 150} = 86.6\,\Omega$.

6.61. By first finding $\bar{\Gamma}(0)$ to be $\frac{2}{3} e^{-j0.6\pi}$ from the standing wave data and following the method of Example 6-25, we obtain the answers given on page 539 of the text.

6.62. From the standing wave data, we get $|\bar{\Gamma}(0)| = \frac{3-1}{3+1} = \frac{1}{2}$ and

$$\underline{/\bar{\Gamma}(0)} = -\pi + \frac{5.80\lambda}{40} \cdot \frac{4\pi}{\lambda} = -0.42\pi \quad \text{so that} \quad \bar{\Gamma}(0) = 0.5\,e^{-j0.42\pi}.$$

Then $$\bar{Y}(0) = \frac{1}{\bar{Z}(0)} = \frac{1}{Z_0}\frac{1-\bar{\Gamma}(0)}{1+\bar{\Gamma}(0)} = 0.02\,\frac{0.75 + j\sin 0.42\pi}{1.25 + \cos 0.42\pi}.$$

If the input susceptance of the stub is B, the effective load admittance with the stub connected is given by

$$\bar{Y}'(0) = \frac{0.02 \times 0.75}{1.25 + \cos 0.42\pi} + j\left[\frac{0.02 \sin 0.42\pi}{1.25 + \cos 0.42\pi} + B\right].$$

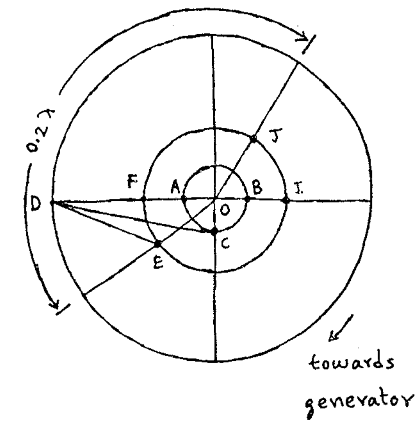
Let this be $C + j(D+B)$. Then

$$\bar{\Gamma}'(0) = \frac{\bar{Z}'(0) - Z_0}{\bar{Z}'(0) + Z_0} = \frac{Y_0 - \bar{Y}'(0)}{Y_0 + \bar{Y}'(0)} = \frac{Y_0 - C - j(D+B)}{Y_0 + C + j(D+B)}.$$

For VSWR $= \dfrac{1+|\bar{\Gamma}'(0)|}{1-|\bar{\Gamma}'(0)|}$ to be a minimum, $\bar{\Gamma}'(0)$ must be a minimum, which occurs for $D+B=0$. Thus for minimum VSWR, $\bar{Y}'(0) = \dfrac{0.02 \times 0.75}{1.25 + \cos 0.42\pi} = \dfrac{1}{100}$ and $\bar{\Gamma}'(0) = \frac{1}{3}$ so that the minimum VSWR that can be achieved is 2.

6.63. Method same as in Example 6-26. See page 539 of the text for answers.

6.64. (a) Normalized load impedance presented to medium 2 is $\dfrac{\eta_0/3}{\eta_0/2} = \dfrac{2}{3}$. We locate this on the Smith chart at point A and draw the constant VSWR circle and read the VSWR value at point B to be 1.5.



(b) Knowing the thickness of medium 2 to be 0.375λ, we go around the constant VSWR circle starting at point A towards the generator by 0.375λ to reach point C and read the normalized input impedance of medium 2 as $0.92 - j0.38$.

(c) To sketch the standing wave pattern in medium 2, we note that we have started at a minimum of $|\bar{E}_x|$ (point A) and have gone through a maximum of $|\bar{E}_x|$ (point B) before reaching point C. If $|\bar{E}_x|$ is assumed to be 1 at the right end of medium 2, then the maximum $|\bar{E}_x|$ is 1.5 at a distance of 2.5 cm and $|\bar{E}_x|$ at the left end of medium 2 is $\dfrac{DC}{DA} = 1.275$.

(d) The normalized load impedance presented to medium 1 is equal to $(0.92 - j0.38)\, \dfrac{\eta_0}{2} \Big/ \eta_0 = (0.46 - j0.19)$. We locate this at point E and draw the constant VSWR circle. The VSWR in medium 1 is then given by the value at point I. This is 2.267.

(e) To sketch the standing wave pattern in medium 1, we start at point E and move around the constant VSWR circle towards the generator. We first reach a minimum of $|\bar{E}_x|$ (point F) at a distance of 0.0367λ or $0.0367 \times 20 = 0.734$ cm. The value of this minimum $|\bar{E}_x|$ is $1.1475 \times 2.267 = 2.6013$. The first maximum occurs at $0.734 + 5$ or 5.734 cm.

(f) Having determined the standing wave pattern for $|\bar{E}_x|$, the standing wave pattern for $|\bar{H}_y|$ can be obtained by knowing that the points for $|\bar{H}_y|$ on the Smith chart are located diagonally opposite to the points for the corresponding $|\bar{E}_x|$. The results are the same as given in the solution for Problem 6.58.

(g) $|\bar{\Gamma}|$ in medium 1 $= \frac{OF}{OD} = 0.39$ which gives the fraction of incident power transmitted into medium 3 as $1 - 0.39^2 = 0.8479$.

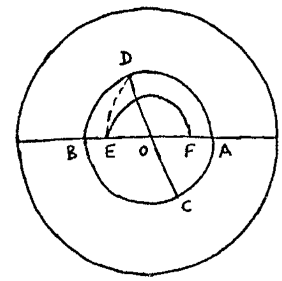
(h) To find the wave impedance in medium 1 at a distance of 4 cm or $\frac{\lambda}{5}$ from the interface between media 1 and 2, we start at point E and move around the constant VSWR circle towards the generator by $\frac{\lambda}{5}$ to reach point J and obtain the required wave impedance as $(1.08 + j0.87)\eta_0$.

6.65. Method is same as in Example 6-27 once we locate the normalized load impedance by following in reverse the procedures illustrated in Example 6-26. For answers, see answers to Problem 6.61 on page 539 of the text.

6.66. (a) Draw constant VSWR ($= 3.0$) circle passing through point A.

(b) Locate point B corresponding to voltage minimum.

(c) Starting at point B, go around the constant VSWR circle towards the load by $\frac{5.80}{40}\lambda = 0.145\lambda$, to locate point C corresponding to the normalized load impedance.

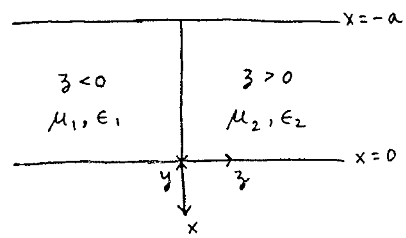


(d) Locate point D corresponding to the normalized load admittance, diametrically opposite to point C.

(e) A stub connected in parallel with the load effectively moves point D along the constant conductance circle passing through that point. The minimum VSWR is then achieved when point D is effectively moved to point E. This minimum VSWR value as read at point F is 2.0.

6.67. Method same as in Example 6-28. See page 539 of the text for answers.

6.68. We first write the expressions for the incident, reflected, and transmitted fields. These are as follows:



Incident fields:

$$\bar{E}_{yi} = -2j\,\bar{E}_{oi}\sin\left(\frac{m\pi x}{a}\right)e^{-j\frac{2\pi}{\lambda_{g1}}z}$$

$$\bar{H}_{xi} = 2j\,\frac{\bar{E}_{oi}}{\eta_1}\frac{\lambda_1}{\lambda_{g1}}\sin\left(\frac{m\pi x}{a}\right)e^{-j\frac{2\pi}{\lambda_{g1}}z}$$

$$\bar{H}_{yi} = 2\,\frac{\bar{E}_{oi}}{\eta_1}\frac{\lambda_1}{\lambda_c}\cos\left(\frac{m\pi x}{a}\right)e^{-j\frac{2\pi}{\lambda_{g1}}z}.$$

Reflected fields:

$$\bar{E}_{yr} = -2j\,\bar{E}_{or}\sin\left(\frac{m\pi x}{a}\right)e^{j\frac{2\pi}{\lambda_{g1}}z}$$

$$\bar{H}_{xr} = -2j\,\frac{\bar{E}_{or}}{\eta_1}\frac{\lambda_1}{\lambda_{g1}}\sin\left(\frac{m\pi x}{a}\right)e^{j\frac{2\pi}{\lambda_{g1}}z}$$

$$\bar{H}_{zr} = 2\,\frac{\bar{E}_{or}}{\eta_1}\frac{\lambda_1}{\lambda_c}\cos\left(\frac{m\pi x}{a}\right)e^{j\frac{2\pi}{\lambda_{g1}}z}$$

Transmitted fields:

$$\bar{E}_{yt} = -2j\,\bar{E}_{ot}\sin\left(\frac{m\pi x}{a}\right)e^{-j\frac{2\pi}{\lambda_{g2}}z}$$

$$\bar{H}_{xt} = 2j\,\frac{\bar{E}_{ot}}{\eta_2}\frac{\lambda_2}{\lambda_{g2}}\sin\left(\frac{m\pi x}{a}\right)e^{-j\frac{2\pi}{\lambda_{g2}}z}$$

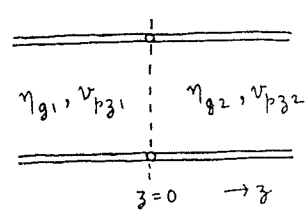
$$\bar{H}_{zt} = 2\,\frac{\bar{E}_{ot}}{\eta_2}\frac{\lambda_2}{\lambda_c}\cos\left(\frac{m\pi x}{a}\right)e^{-j\frac{2\pi}{\lambda_{g2}}z}$$

Now, using the boundary conditions at $z=0$ given by

$\bar{E}_{yi}+\bar{E}_{yr}=\bar{E}_{yt}$, and $\bar{H}_{xi}+\bar{H}_{xr}=\bar{H}_{xt}$, we get

$$\frac{\bar{E}_{or}}{\bar{E}_{oi}} = \frac{\eta_2\frac{\lambda_{g2}}{\lambda_2}-\eta_1\frac{\lambda_{g1}}{\lambda_1}}{\eta_2\frac{\lambda_{g2}}{\lambda_2}+\eta_1\frac{\lambda_{g1}}{\lambda_1}} = \frac{\eta_{g2}-\eta_{g1}}{\eta_{g2}+\eta_{g1}}$$

$$\frac{\bar{E}_{ot}}{\bar{E}_{oi}} = \frac{2\eta_2\frac{\lambda_{g2}}{\lambda_2}}{\eta_2\frac{\lambda_{g2}}{\lambda_2}+\eta_1\frac{\lambda_{g1}}{\lambda_1}} = \frac{2\eta_{g2}}{\eta_{g2}+\eta_{g1}}$$



which give the transmission line equivalent shown in the accompanying figure.

6.69. By starting with the expressions for $\bar{H}_i$, $\bar{E}_i$, $\bar{H}_r$, and $\bar{E}_r$ as given by

$$\bar{H}_i = \bar{H}_0 e^{-j(\beta x \cos\theta_i + \beta z \sin\theta_i)} \mathbf{i}_y$$

$$\bar{E}_i = \sqrt{\frac{\mu}{\epsilon}} [\bar{H}_0 \sin\theta_i \mathbf{i}_x - \bar{H}_0 \cos\theta_i \mathbf{i}_z] e^{-j(\beta x \cos\theta_i + \beta z \sin\theta_i)}$$

$$\bar{H}_r = \bar{H}_0' e^{j(\beta x \cos\theta_r - \beta z \sin\theta_r)} \mathbf{i}_y$$

$$\bar{E}_r = \sqrt{\frac{\mu}{\epsilon}} [\bar{H}_0' \sin\theta_r \mathbf{i}_x + \bar{H}_0' \cos\theta_r \mathbf{i}_z] e^{j(\beta x \cos\theta_r - \beta z \sin\theta_r)}$$

and proceeding in a manner similar to the treatment for TE waves in Section 6-12, we obtain the results given on page 539 of the text.

6.70. $\eta_{g1} = \eta_1\sqrt{1-(\lambda_1/\lambda_c)^2} = 208.4\,\Omega$, $\eta_{g2} = \eta_2\sqrt{1-(\lambda_2/\lambda_c)^2} = 171.36\,\Omega$

$$\bar{\Gamma} = \frac{\eta_{g2} - \eta_{g1}}{\eta_{g2} + \eta_{g1}} = -0.097535.$$

Fraction of incident power transmitted into region $z > 0$ $= 1 - |\bar{\Gamma}|^2$ $= 0.990487$. From $\eta_{g3} = \eta_3\sqrt{1-(\lambda_3/\lambda_c)^2} = \eta_{g1}\eta_{g2}$, we obtain $\epsilon_3 = 0.9172\,\epsilon_0$ or $3.083\,\epsilon_0$. Ruling out $0.9172\,\epsilon_0$, we have $\epsilon_3 = 3.083\,\epsilon_0$. Then finding λ_{g3} and dividing by 4, we get the required length of the matching section to be 0.80875 cm.

6.71. From $\beta_z = \dfrac{\omega}{v_{pz}}$, we have

$$\frac{d\beta_z}{d\omega} = \frac{v_{pz} - \omega \dfrac{dv_{pz}}{d\omega}}{v_{pz}^2} \quad \text{and} \quad v_{gz} = \frac{d\omega}{d\beta_z} = \frac{v_{pz}}{1 - \dfrac{\omega}{v_{pz}}\dfrac{dv_{pz}}{d\omega}}.$$

6.72. Using $v_{gz} = v_p\sqrt{1-\left(\frac{\lambda}{\lambda_c}\right)^2}$, we obtain the following values:

Mode	$TE_{1,0}$	$TE_{2,0}$	$TE_{3,0}$
v_{gz}, m/sec	1.4256×10^8	1.1709×10^8	0.522×10^8

6.73. (a) From $\dfrac{\partial \bar{V}}{\partial z} = -j\omega \mathcal{L}\, \bar{I}(z)$ and $\dfrac{\partial \bar{I}}{\partial z} = -j\omega \mathcal{C}\, \bar{V}(z)$,

$$\frac{\partial^2 \bar{V}}{\partial z^2} = -j\omega \frac{\partial \mathcal{L}}{\partial z}\bar{I}(z) - j\omega\mathcal{L}\frac{\partial \bar{I}(z)}{\partial z} = -j\omega\frac{\partial \mathcal{L}}{\partial z}\left(-\frac{1}{j\omega\mathcal{L}}\right)\frac{\partial \bar{V}}{\partial z} - j\omega\mathcal{L}[-j\omega\mathcal{C}\bar{V}(z)]$$

$$\text{or} \quad \frac{\partial^2 \bar{V}}{\partial z^2} - \frac{1}{\mathcal{L}}\left(\frac{\partial \mathcal{L}}{\partial z}\right)\left(\frac{\partial \bar{V}}{\partial z}\right) + \omega^2\mathcal{L}\mathcal{C}\bar{V} = 0$$

$$\frac{\partial^2 \bar{I}}{\partial z^2} = -j\omega\frac{\partial \mathcal{C}}{\partial z}\bar{V}(z) - j\omega\mathcal{C}\frac{\partial \bar{V}(z)}{\partial z} = -j\omega\frac{\partial \mathcal{C}}{\partial z}\left(-\frac{1}{j\omega\mathcal{C}}\right)\frac{\partial \bar{I}}{\partial z} - j\omega\mathcal{C}[-j\omega\mathcal{L}\bar{I}(z)]$$

$$\text{or} \quad \frac{\partial^2 \bar{I}}{\partial z^2} - \frac{1}{\mathcal{C}}\left(\frac{\partial \mathcal{C}}{\partial z}\right)\left(\frac{\partial \bar{I}}{\partial z}\right) + \omega^2\mathcal{L}\mathcal{C}\bar{I} = 0$$

(b) For $\mathcal{L} = \mathcal{L}_0 e^{-az}$ and $\mathcal{C} = \mathcal{C}_0 e^{az}$, the equations become

$$\frac{\partial^2 \bar{V}}{\partial z^2} + a\frac{\partial \bar{V}}{\partial z} + \omega^2 \mathcal{L}_0 \mathcal{C}_0 \bar{V} = 0 \quad \text{and} \quad \frac{\partial^2 \bar{I}}{\partial z^2} - a\frac{\partial \bar{I}}{\partial z} + \omega^2 \mathcal{L}_0 \mathcal{C}_0 \bar{I} = 0$$

for which the solutions are given on page 539 of the text.

Real exponents in the solutions for $\bar{V}$ and $\bar{I}$ mean no propagation.

This occurs for $a^2 - 4\omega^2 \mathcal{L}_0 \mathcal{C}_0 > 0$ and hence the cut off frequency is given by $f_c = \dfrac{a}{4\pi\sqrt{\mathcal{L}_0 \mathcal{C}_0}}$.

6.74. The required expression is $\alpha\lambda = 2\pi \left[\dfrac{\sqrt{1+(\sigma^2/\omega^2\epsilon^2)} - 1}{\sqrt{1+(\sigma^2/\omega^2\epsilon^2)} + 1} \right]^{1/2}$

6.75. For $f = 100$ MHz, $\sigma \ll \omega\epsilon$ and for $f = 10$ kHz, $\sigma \ll \omega\epsilon$. See page 540 of the text for answers.

6.76. Since the slab is of infinite depth, the input impedance of the medium is equal to its intrinsic impedance $\bar{\eta} = (1+j)\sqrt{\dfrac{\pi f \mu}{\sigma}}$. Thus

$$\bar{\Gamma} = \frac{\bar{\eta} - \eta_0}{\bar{\eta} + \eta_0} = \frac{(\sqrt{\pi f \mu_0/\sigma} - \eta_0) + j\sqrt{\pi f \mu_0/\sigma}}{(\sqrt{\pi f \mu_0/\sigma} + \eta_0) + j\sqrt{\pi f \mu_0/\sigma}}$$

Writing the expression for $|\bar{\Gamma}|^2$ and using the condition $\dfrac{\sigma}{\omega\epsilon_0} \gg 1$, we get the fraction of the incident power reflected $\approx 1 - 4\sqrt{\dfrac{\pi f \epsilon_0}{\sigma}}$ and the fraction of the incident power transmitted into the conductor $= 1 - |\bar{\Gamma}|^2 \approx 4\sqrt{\dfrac{\pi f \epsilon_0}{\sigma}}$. For copper at $f = 30$ MHz, These two quantities are $1 - 15.16 \times 10^{-6}$ and 15.16×10^{-6}, respectively.

6.77. (a) See page 540 of the text.

(b) $$\bar{Z}(d) = \frac{\bar{E}_x(d)}{\bar{H}_y(d)} = \bar{\eta}\,\frac{1 + \dfrac{\bar{E}_x^-}{\bar{E}_x^+} e^{-2\bar{\gamma} d}}{1 - \dfrac{\bar{E}_x^-}{\bar{E}_x^+} e^{-2\bar{\gamma} d}} = \bar{\eta}\,\frac{1 + \bar{\Gamma}(0) e^{-2\bar{\gamma} d}}{1 - \bar{\Gamma}(0) e^{-2\bar{\gamma} d}}$$

(c) Since the impedance seen just to the right of the lossy conductor is ∞, the reflection coefficient there is equal to 1. Then the reflection coefficient at the left end of the lossy conductor is $1 e^{-2\bar{\gamma}_c t}$ where $\bar{\gamma}_c$ is the propagation constant in the lossy conductor. Thus the impedance presented to the wave incident from free space onto the lossy conductor is

$$\bar{Z}_L = \bar{\eta}_c \frac{1 + 1\, e^{-2\bar{\gamma}_c t}}{1 - 1\, e^{-2\bar{\gamma}_c t}} = \bar{\eta}_c \coth \bar{\gamma}_c t$$

For $\alpha_c t \ll 1$, $\beta_c t$ is also $\ll 1$ since $\beta_c = \alpha_c$, and

$$\bar{Z}_L \approx \frac{\bar{\eta}_c}{\bar{\gamma}_c t} = (1+j)\sqrt{\frac{\pi f \mu}{\sigma}} \frac{1}{(1+j)\sqrt{\pi f \mu \sigma}\; t} = \frac{1}{\sigma t}.$$

If $\frac{1}{\sigma t} = \eta_0$, that is, $\sigma = \frac{1}{\eta_0 t}$, there will be no reflection of the waves.

6.78. Substituting $\bar{E}_{x0} = \bar{\eta}\, \bar{H}_{y0} = \eta \frac{\bar{I}_x}{w} = (1+j)\sqrt{\frac{\pi f \mu}{\sigma}} \frac{\bar{I}_x}{w}$ in

Equations (6-287a) and (6-278b), we get

$$|\bar{E}_x| = \sqrt{\frac{2\pi f \mu}{\sigma}} \frac{|\bar{I}_x|}{w} e^{-\sqrt{\pi f \mu \sigma}\, z} \quad \text{and} \quad |\bar{H}_y| = \frac{|\bar{I}_x|}{w} e^{-\sqrt{\pi f \mu \sigma}\, z}$$

(a) $$\int_v \frac{1}{2} \sigma |\bar{E}_x|^2 \, dv = \frac{1}{2}\sigma \int_{z=0}^{\infty} \int_{y=y}^{y+w} \int_{x=x}^{x+l} \frac{2\pi f \mu}{\sigma} \frac{|\bar{I}_x|^2}{w^2} e^{-2\sqrt{\pi f \mu \sigma}\, z} \, dx\, dy\, dz$$

$$= \frac{1}{2} \frac{l}{\sigma \delta w} |\bar{I}_x|^2 .$$

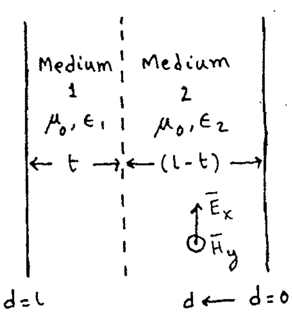
(b) $$\int_v \frac{1}{4} \mu |\bar{H}_y|^2 \, dv = \frac{1}{4}\mu \int_{z=0}^{\infty} \int_{y=y}^{y+w} \int_{x=x}^{x+l} \frac{|\bar{I}_x|^2}{w^2} e^{-2\sqrt{\pi f \mu \sigma}\, z} \, dx\, dy\, dz$$

$$= \frac{1}{2w}\left[\frac{1}{2} \frac{l}{\sigma \delta w} |\bar{I}_x|^2\right].$$

6.79. See page 540 of the text.

6.80. The electric field at a time at which the magnetic field is zero everywhere is $-2E_0 \sin \frac{n\pi d}{l}$. Then, the required total energy density from $d=0$ to $d=l$ is $\int_{d=0}^{l} \frac{1}{2}\epsilon \left(-2E_0 \sin \frac{n\pi d}{l}\right)^2 dd = \epsilon E_0^2 l$ which is the same as the result given by Equation (6-299).

6.81. (a) Writing the expressions for the fields in the two regions, we have for medium 2,



$$\bar{E}_{x2} = 2j\, \bar{E}_{02} \sin \beta_2 d$$

$$\bar{H}_{y2} = 2 \frac{\bar{E}_{02}}{\eta_2} \cos \beta_2 d$$

and for medium 1,

$$\bar{E}_{x1} = 2j\, \bar{E}_{01} \sin \beta_1 (d-l)$$

$$\bar{H}_{y1} = 2 \frac{\bar{E}_{01}}{\eta_1} \cos \beta_1 (d-l)$$

Now using the boundary conditions

$$[\bar{E}_{x1}]_{d=l-t} = [\bar{E}_{x2}]_{d=l} \quad \text{and} \quad [\bar{H}_{y1}]_{d=l-t} = [\bar{H}_{y2}]_{d=l}$$

we obtain $-2j\,\bar{E}_{01}\sin\beta_1 t = 2j\,\bar{E}_{02}\sin\beta_2(l-t)$

$$2\,\frac{\bar{E}_{01}}{\eta_1}\cos\beta_1 t = 2\,\frac{\bar{E}_{02}}{\eta_2}\cos\beta_2(l-t)$$

Dividing one equation by the other and rearranging, we get the required result.

(b) For the given values of t, l, ϵ_1, and ϵ_2, the equation of part (a) gives $\tan \omega\sqrt{\mu_0\epsilon_0}\,\frac{l}{2} = 2$. For values of frequencies, see page 540 of the text.

6.82. For a particular mode of operation, that is, for fixed n, $l = \dfrac{n}{2 f_n\sqrt{\mu\epsilon}}$

and $Q = \dfrac{l}{2\delta} = \dfrac{n}{2 f_n\sqrt{\mu\epsilon}}\sqrt{\pi f_n \mu\sigma} \propto \dfrac{1}{\sqrt{f_n}}$.

6.83. (a) We first write expressions for (+) and (−) wave fields corresponding to $TE_{m,0}$ modes in parallel-plate guide. These are given by

$$\bar{E}_y^+ = -2j\,\bar{E}_0 \sin\left(\frac{m\pi x}{a}\right) e^{-j\frac{2\pi}{\lambda_g}z}$$

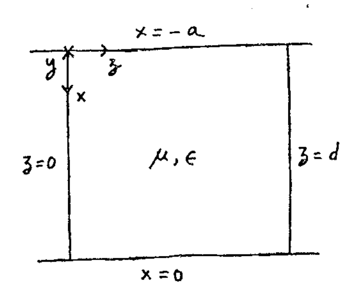
$$\bar{H}_x^+ = 2j\,\frac{\bar{E}_0}{\eta}\frac{\lambda}{\lambda_g}\sin\left(\frac{m\pi x}{a}\right) e^{-j\frac{2\pi}{\lambda_g}z}$$

$$\bar{H}_z^+ = 2\,\frac{\bar{E}_0}{\eta}\frac{\lambda}{\lambda_c}\cos\left(\frac{m\pi x}{a}\right) e^{-j\frac{2\pi}{\lambda_g}z}$$

$$\bar{E}_y^- = -2j\,\bar{E}_0'\sin\left(\frac{m\pi x}{a}\right) e^{j\frac{2\pi}{\lambda_g}z}$$

$$\bar{H}_x^- = -2j\,\frac{\bar{E}_0'}{\eta}\frac{\lambda}{\lambda_g}\sin\left(\frac{m\pi x}{a}\right) e^{j\frac{2\pi}{\lambda_g}z}$$

$$\bar{H}_z^- = 2\,\frac{\bar{E}_0'}{\eta}\frac{\lambda}{\lambda_c}\cos\left(\frac{m\pi x}{a}\right) e^{j\frac{2\pi}{\lambda_g}z}$$



To obtain the expressions for complete standing waves, we now set $\bar{E}_0' = -\bar{E}_0$ and add the two sets of expressions. Thus, absorbing the factor 4 into the constant, we obtain

$$\bar{E}_y = \bar{E}_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{2\pi}{\lambda_g}z\right)$$

$$\bar{H}_x = -j\,\frac{\bar{E}_0}{\eta}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{2\pi}{\lambda_g}z\right)$$

$$\bar{H}_z = j\,\frac{\bar{E}_0}{\eta}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{2\pi}{\lambda_g}z\right)$$

$\bar{E}_y$ has nodes at $z=0$ and at intervals in z which are integer multiples of $\frac{\lambda_g}{2}$. Hence, perfect conductors occupying the planes $z=0$ and $z=d$ support standing waves of guide wavelengths $\lambda_{gn} = \frac{2d}{l}$, $l = 1,2,3,\ldots$

Substituting $\lambda_{gn} = \lambda_n\Big/\sqrt{1-\left(\frac{\lambda_n}{\lambda_c}\right)^2}$ and simplifying, we get

$$\lambda_n = \frac{1}{\sqrt{\left(\frac{l}{2d}\right)^2 + \left(\frac{m}{2a}\right)^2}}$$ which gives $f_n = f_{m,0,l} = \frac{1}{2\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{l}{d}\right)^2}$.

These are called the $TE_{m,0,l}$ modes since the fields have m half-sinusoidal variations in the x direction, no variations in the y direction and l half sinusoidal variations in the z direction. See page 540 of the text for the lowest three resonant frequencies and the corresponding mode numbers for $a = d = 4$ cm.

(b) For $TE_{1,0,1}$ mode, $m = 1$, $\lambda_c = 2a$ and $\lambda_g = 2d$. Substituting these in the expressions for the total fields found in part (a), we get the expressions given on page 540 of the text.

To find the Q of the resonator, we first find the energy stored in the resonator per unit length in the y direction. For the $TE_{1,0,1}$ mode, this is given by

$$W = \int_{x=-a}^{0}\int_{z=0}^{d} \frac{1}{2}\epsilon E_0^2 \sin^2\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{\pi z}{d}\right)\,dx\,dz = \frac{1}{8}\,ad\,E_0^2.$$

We next find the power dissipated in the walls of the resonator per unit length in the y direction. To do this, we note that

$$[\bar{J}_y]_{z=0} = [\bar{H}_x]_{z=0} = -j\frac{\bar{E}_0}{\eta}\frac{\lambda}{2d}\sin\frac{\pi x}{a}$$

$$[\bar{J}_y]_{z=d} = -[\bar{H}_x]_{z=d} = -j\frac{\bar{E}_0}{\eta}\frac{\lambda}{2d}\sin\frac{\pi x}{a}$$

$$[\bar{J}_y]_{x=0} = [\bar{H}_z]_{x=0} = j\frac{\bar{E}_0}{\eta}\frac{\lambda}{2a}\sin\frac{\pi z}{d}$$

$$[\bar{J}_y]_{x=-a} = -[\bar{H}_z]_{x=-a} = j\frac{\bar{E}_0}{\eta}\frac{\lambda}{2a}\sin\frac{\pi z}{d}$$

Considering the wall $z = 0$, the current flowing in a width dx at an arbitrary value of x is $\bar{I}_y = [\bar{J}_y]_{z=0}\,dx$ $= -j\frac{\bar{E}_0}{\eta}\frac{\lambda}{2d}\sin\frac{\pi x}{a}\,dx$. The resistance offered to this current flow per unit length in the y direction is $R_s = \frac{1}{\sigma\delta\,(dx)}$. The power dissipated in the wall per unit length in the y direction is

$$[P_d]_{z=0} = \int_{x=-a}^{0}\frac{1}{2}\,\frac{1}{\sigma\delta(dx)}\left(\frac{|\bar{E}_0|}{\eta}\frac{\lambda}{2d}\sin\frac{\pi x}{a}\,dx\right)^2 = \frac{|\bar{E}_0|^2\lambda^2}{8\sigma\delta\eta^2 d^2}\,\frac{a}{2}.$$

Similarly, finding $[P_d]_{z=d}$, $[P_d]_{x=0}$, and $[P_d]_{x=-a}$, we obtain the total power dissipated per unit length in the y direction as

$$P_d = \frac{|\bar{E}_0|^2 \lambda^2}{8\sigma\delta\eta^2}\left[\frac{a}{d^2} + \frac{d}{a^2}\right].$$

The Q of the resonator is then given by

$$Q = 2\pi f \frac{W}{P_d} = 2\pi f \frac{\epsilon a d \sigma \delta \eta^2}{\lambda^2} \frac{a^2 d^2}{(a^3 + d^3)}.$$

But $\dfrac{f}{\lambda^2} = \dfrac{1}{2\sqrt{\mu\epsilon}}\sqrt{\left(\frac{1}{d}\right)^2 + \left(\frac{1}{a}\right)^2} \cdot \frac{1}{4}\left[\left(\frac{1}{d}\right)^2 + \left(\frac{1}{a}\right)^2\right] = \dfrac{1}{8\sqrt{\mu\epsilon}}\left[\dfrac{a^2+d^2}{a^2 d^2}\right]^{3/2}.$

Substituting into the expression for Q, we get

$$Q = \frac{\pi\sigma\delta\eta}{4}\frac{(a^2+d^2)^{3/2}}{a^3+d^3}.$$

6.84. (a) $W = \epsilon E_0^2 l$

$$P_d = \int_{d=0}^{l} p_d \, dd = \int_{d=0}^{l} \frac{1}{2}\sigma_d |\bar{E}_x|^2 \, dd = \int_{d=0}^{l} 2\sigma_d E_0^2 \sin^2\frac{n\pi d}{l} \, dd = \sigma_d E_0^2 l.$$

$$Q_1 = 2\pi f \frac{W}{P_d} = \frac{\omega\epsilon}{\sigma_d}.$$

(b) $P_d = \sigma_d E_0^2 l + 4\epsilon E_0^2 \sqrt{\dfrac{\pi f}{\mu\sigma}}$

$$Q = 2\pi f \frac{\epsilon E_0^2 l}{\sigma_d E_0^2 l + 4\epsilon E_0^2\sqrt{\frac{\pi f}{\mu\sigma}}}$$

or $\dfrac{1}{Q} = \dfrac{\sigma_d}{\omega\epsilon} + \dfrac{2\delta}{l} = \dfrac{1}{Q_1} + \dfrac{1}{Q_2}.$

6.85. $\sqrt{\dfrac{Ne^2}{m\epsilon_0}} \rightarrow \left[\dfrac{(m)^{-3}\ (\text{coulomb})^2}{(\text{kg})\ (\text{coulomb})^2/(\text{newton})(\text{meter})^2}\right]^{1/2}$

$$\rightarrow \left[\frac{\text{newton}}{(\text{m})(\text{kg})}\right]^{1/2} \rightarrow \left[\frac{\text{m}\ (\text{sec})^{-2}}{\text{m}}\right]^{1/2} \rightarrow (\text{sec})^{-1}.$$

$$\frac{e^2}{4\pi^2 m\epsilon_0} = \frac{1.6^2 \times 10^{-38} \times 36\pi}{4\pi^2 \times 9.1 \times 10^{-31} \times 10^{-9}} = 80.6$$

6.86. (a) A wave of frequency f incident obliquely on the ionosphere at an angle θ_0 with the normal is reflected from a level at which the plasma frequency is equal to $f\cos\theta_0$. Hence the required value of θ_0 is given by $10 = 20\cos\theta_0$ or $\theta_0 = 60°$.

(b) $10 = f\cos\theta_0 = f\cos 30°$ or $f = \frac{20}{\sqrt{3}} = 11.547$ MHz.

6.87. For $f = 20$ MHz and $f_N = 12$ MHz, $\mu = \sqrt{1 - \frac{f_N^2}{f^2}} = 0.8$. If θ is the angle which the path of the wave makes with the vertical in the slab ionosphere, then $0.8 \sin\theta = \sin 30^\circ = 0.5$ or $\theta = 38^\circ 41'$. Then the horizontal range of the satellite from the receiver is

$100 \tan 30^\circ + 400 \tan 38^\circ 41' + 500 \tan 30^\circ = 666.678$ km.

The true elevation angle of the satellite is $\tan^{-1} \frac{1000}{666.678} = 56.3^\circ$.

6.88. Time delay undergone by the satellite signal is given by

$$T_g = \int_A^S \frac{ds}{v_g} = \int_A^S \frac{ds}{c\sqrt{1 - \frac{f_N^2}{f^2}}} \approx \int_A^S \frac{ds}{c}\left[1 + \frac{f_N^2}{2f^2}\right]$$

$$= \int_A^S \frac{ds}{c} + \int_A^S \frac{f_N^2}{2cf^2}\, ds$$

Apparent range of the satellite is given by

$$R' = cT_g = \int_A^S ds + \int_A^S \frac{f_N^2}{2f^2}\, ds = \int_A^S ds + \frac{40.3}{f^2}\int_A^S N\, ds.$$

Thus the excess range is $\frac{40.3}{f^2}\int_A^S N\, ds$. For $\int_A^S N\, ds = 10^{18}$ electrons/m^2, the excess range values are 2.056 km and 15.74 m for 140 MHz and 1600 MHz, respectively.

6.89. (a) At a point along the x axis, the field due to dipole 1 (along the x axis) is zero. The field due to dipole 2 (along the z axis) has only a z component.

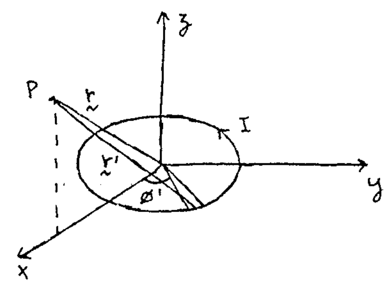
(b) At a point along the z axis, the field due to dipole 1 has only an x component. The field due to dipole 2 is zero.

(c) At a point along the y axis, the field due to dipole 1 has only an x component and the field due to dipole 2 has only a z component. The two fields are equal in magnitude.

(d) At a point along the line $x = 0$, $y = z$, the field due to dipole 1 is in the direction of $-\mathbf{i}_x$ and the field due to dipole 2 is in the direction of $\mathbf{i}_\theta = \frac{1}{\sqrt{2}}(\mathbf{i}_y - \mathbf{i}_z)$ at that point. The magnitude of the field due to dipole 2 is $\frac{1}{\sqrt{2}}$ times the magnitude of the field due to dipole 1.

For descriptions of polarization of the field for each case, see page 540 of the text.

6.90. (a) From symmetry considerations, the vector potential due to the loop is entirely in the ϕ direction. Hence, considering a point P in the xy plane, we have



$$\underset{\sim}{A} = \int_{\phi'=0}^{2\pi} \frac{\mu_0 I_0 a \, d\phi' \cos \omega\left(t - \frac{|\underset{\sim}{r}-\underset{\sim}{r}'|}{v}\right)}{4\pi |\underset{\sim}{r}-\underset{\sim}{r}'|} (-\sin\phi' \, \underset{\sim}{i}_x + \cos\phi' \, \underset{\sim}{i}_y)$$

$$\bar{\underset{\sim}{A}} = \int_{\phi'=0}^{2\pi} \frac{\mu_0 I_0 a \, d\phi' \, e^{-j\omega \frac{|\underset{\sim}{r}-\underset{\sim}{r}'|}{v}}}{4\pi |\underset{\sim}{r}-\underset{\sim}{r}'|} (-\sin\phi' \, \underset{\sim}{i}_x + \cos\phi' \, \underset{\sim}{i}_y)$$

But $|\underset{\sim}{r}-\underset{\sim}{r}'| = [(x - a\cos\phi')^2 + (-a\sin\phi')^2 + z^2]^{1/2}$

$$\approx r\left(1 - \frac{a}{r}\sin\theta \cos\phi'\right) \quad \text{for } r \gg a$$

and $\dfrac{1}{|\underset{\sim}{r}-\underset{\sim}{r}'|} \approx \dfrac{1}{r}\left(1 + \dfrac{a}{r}\sin\theta\cos\phi'\right)$

$$e^{-j\omega \frac{|\underset{\sim}{r}-\underset{\sim}{r}'|}{v}} \approx e^{-j\omega\frac{r}{v}} e^{j\frac{\omega}{v} a \sin\theta \cos\phi'}$$

$$\approx e^{-j\omega\frac{r}{v}}\left(1 + j\frac{\omega}{v} a\sin\theta\cos\phi'\right) \quad \text{for } \frac{\omega}{v}a = \beta a = \frac{2\pi a}{\lambda} \ll 1$$

Thus

$$\bar{\underset{\sim}{A}} \approx \frac{\mu_0 I a}{4\pi r} e^{-j\omega\frac{r}{v}} \int_{\phi'=0}^{2\pi} \left(1 + j\frac{\omega}{v}\sin\theta\cos\phi'\right)\cdot\left(1 + \frac{a}{r}\sin\theta\cos\phi'\right)\cdot$$
$$(-\sin\phi' \, \underset{\sim}{i}_x + \cos\phi' \, \underset{\sim}{i}_y)\, d\phi'$$

$$\approx \frac{\mu_0 I a}{4\pi r} e^{-j\omega\frac{r}{v}} \int_{\phi'=0}^{2\pi} \left(1 + j\frac{\omega}{v} a\sin\theta\cos\phi' + \frac{a}{r}\sin\theta\cos\phi'\right)\cdot$$
$$(-\sin\phi' \, \underset{\sim}{i}_x + \cos\phi' \, \underset{\sim}{i}_y)\, d\phi'$$

where we have neglected the term $j\frac{\omega}{v}\frac{a^2}{r}\sin^2\theta\cos^2\phi$ since both $\frac{\omega a}{v}$ and $\frac{a}{r}$ are $\ll 1$. Evaluating the integrals and replacing $\underset{\sim}{i}_y$ by by $\underset{\sim}{i}_\phi$, we obtain finally

$$\bar{\underset{\sim}{A}} \approx \frac{\mu_0 I_0 \pi a^2 \sin\theta}{4\pi r}\left(j\frac{\omega}{v} + \frac{1}{r}\right) e^{j\omega\left(t - \frac{r}{v}\right)} \underset{\sim}{i}_\phi .$$

(b) Using $\bar{H} = \frac{1}{\mu_0}\nabla \times \bar{A}$ and then $\bar{E} = \frac{1}{j\omega\epsilon}\nabla \times \bar{H}$, we get

$$\bar{H} = \frac{I_0 \pi a^2}{4\pi r}\left[\frac{2}{r}\left(j\frac{\omega}{v} + \frac{1}{r}\right)\cos\theta\, \mathbf{i}_r + \left(\frac{1}{r^2} + j\frac{\omega}{vr} - \frac{\omega^2}{v^2}\right)\sin\theta\, \mathbf{i}_\theta\right] e^{-j\omega\frac{r}{v}}$$

$$\bar{E} = \frac{\eta_0 I_0 \pi a^2 \sin\theta}{4\pi r}\left(\frac{\omega^2}{v^2} - j\frac{\omega}{vr}\right) e^{-j\omega\frac{r}{v}}\, \mathbf{i}_\phi .$$

(c) The radiation fields are the fields corresponding to $r \ll \lambda$ or $\frac{\omega}{v} \gg \frac{1}{r}$. Thus we obtain the expressions given in the problem.

6.91. (a) See page 540 of the text.

(b) $$\mathbf{A} = \left[\frac{\mu\omega Q_0 \cos\omega\left(t - \frac{r_1}{v}\right)\cdot dl}{4\pi r_1} - \frac{\mu\omega Q_0 \cos\omega\left(t - \frac{r_2}{v}\right)\cdot dl}{4\pi r_2}\right]\mathbf{i}_z$$

substituting $r_1 \approx r - \frac{dl}{2}\cos\theta$ and $r_2 \approx r + \frac{dl}{2}\cos\theta$ and considering the limit $dl \to 0$, keeping $Q_0(dl)^2$ constant, we get

$$\mathbf{A} = \frac{\mu\omega Q_0 (dl)^2\cos\theta}{4\pi r}\left[-\frac{\omega\sin\omega\left(t - \frac{r}{v}\right)}{v} + \frac{\cos\omega\left(t - \frac{r}{v}\right)}{r}\right]\mathbf{i}_z$$

which corresponds to $\bar{\mathbf{A}}$ given on page 540 of the text.

(c) Using $\bar{H} = \frac{1}{\mu}\nabla \times \bar{A}$ and then $\bar{E} = \frac{1}{j\omega\epsilon}\nabla \times \bar{H}$, we obtain the expressions for $\bar{\mathbf{E}}$ and $\bar{\mathbf{H}}$ given on page 540 of the text.

(d) See page 540 of the text for expressions for the radiation fields which are obtained by using the condition $\frac{\omega}{v} \gg \frac{1}{r}$. To obtain directly from the radiation fields due to the oscillating dipole, we write

$$\bar{E}_\theta = \frac{j\beta\eta\omega Q_0\, dl\sin\theta}{4\pi r_1} e^{-j\beta r_1} - \frac{j\beta\eta\omega Q_0\, dl\sin\theta}{4\pi r_2} e^{-j\beta r_2}$$

$$\bar{H}_\phi = \frac{j\beta\omega Q_0\, dl\sin\theta}{4\pi r_1} e^{-j\beta r_1} - \frac{j\beta\omega Q_0\, dl\sin\theta}{4\pi r_2} e^{-j\beta r_2}$$

Setting $r_1 \approx r - \frac{dl}{2}\cos\theta$ and $r_2 \approx r + \frac{dl}{2}\cos\theta$ and considering the limit $dl \to 0$, keeping $Q_0(dl)^2$ constant, we get the expressions for the radiation fields due to the oscillating quadrupole.

6.92. For 100 MHz, $R_{rad} = 80\pi^2\left(\frac{dl}{\lambda}\right)^2 = 80\pi^2\left(\frac{1}{300}\right)^2 = 0.00871\,\Omega$ and

$\delta = \frac{1}{\sqrt{\pi f\mu\sigma}} = 0.0066$ mm, $R_{ohmic} = \frac{1}{2\pi a}\frac{l}{\sigma\delta} = \frac{1}{2\pi\times 10^{-3}}\frac{10^{-2}}{5.8\times 10^7 \times 0.0066\times 10^{-3}}$

$= 0.004158\,\Omega$. For 300 MHz, $R_{rad} = 0.07894\,\Omega$, $R_{ohmic} = 0.007203\,\Omega$.

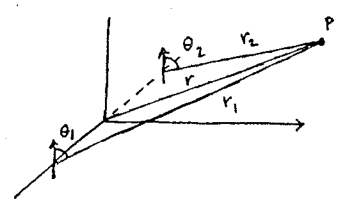
6.93. (a) and (b) Method same as that in Example 6-34.

(c) and (d) Method same as that used for the Hertzian and short dipoles following Example 6-34. see page 540 of the text for answer to (c).

6.94. The radiation fields at point P due to the array of the two short dipoles are given by

$$\bar{E}_\theta = \frac{j\beta\eta L \bar{I}_0}{8\pi}\left[\frac{\sin\theta_1\, e^{-j\beta r_1}}{r_1} + \frac{\sin\theta_2\, e^{-j\beta r_2}}{r_2}\right]$$

$$\bar{H}_\phi = \frac{j\beta L \bar{I}_0}{8\pi}\left[\frac{\sin\theta_1\, e^{-j\beta r_1}}{r_1} + \frac{\sin\theta_2\, e^{-j\beta r_2}}{r_2}\right]$$



Since $r \gg d$, we can replace θ_1, θ_2, r_1, and r_2 in the amplitude factors by θ and r. For r_1 and r_2 in the phase factors, we write

$$r_1 = \left[(x-\tfrac{d}{2})^2 + y^2 + z^2\right]^{1/2} \approx r\left[1 - \frac{d}{2r}\sin\theta\cos\phi\right]$$

$$r_2 = \left[(x+\tfrac{d}{2})^2 + y^2 + z^2\right]^{1/2} \approx r\left[1 + \frac{d}{2r}\sin\theta\cos\phi\right]$$

Thus we obtain

$$\bar{E}_\theta = \frac{j\beta\eta L I_0 \sin\theta}{8\pi r}\, e^{-j\beta r}\cdot 2\cos\left(\frac{\beta d\sin\theta\cos\phi}{2}\right)$$

$$\bar{H}_\phi = \frac{j\beta L I_0 \sin\theta}{8\pi r}\, e^{-j\beta r}\cdot 2\cos\left(\frac{\beta d\sin\theta\cos\phi}{2}\right)$$

Finding $\langle \mathbf{P}_{rad}\rangle = \frac{1}{2}\,\mathrm{Re}\,[\bar{\mathbf{E}}\times\bar{\mathbf{H}}^*]$ and then $U = \langle \mathbf{P}_{rad}\rangle \cdot r^2\,\mathbf{i}_r$,

we get $U = \dfrac{\eta\beta^2 L^2 |\bar{I}_0|^2}{32\pi^2}\sin^2\theta\,\cos^2\left(\dfrac{\beta d\sin\theta\cos\phi}{2}\right)$ and

$$U_n = \frac{U}{U_{max}} = \sin^2\theta\cdot\cos^2\left(\frac{\beta d\sin\theta\cos\phi}{2}\right).$$

For $d = \frac{\lambda}{2}$,

$U_n = \sin^2\theta$ for $\phi = \frac{\pi}{2}$

$U_n = \sin^2\theta\cdot\cos^2\left(\frac{\pi}{2}\sin\theta\right)$ for $\phi = 0$

$U_n = \cos^2\left(\frac{\pi}{2}\cos\phi\right)$ for $\theta = \frac{\pi}{2}$.