Fundamentals of Electromagnetics for Electrical and Computer Engineering in 108 Slides: A Tutorial

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Om Amriteswaryai Namaha!
Om Namah Sivaya!

Inscription on wristband from
Retreat with Amma Mata Amritanandamayi Devi:
“Fill your heart with love and then express it in everything you do.”
In this publication, I have attempted it through 108 slides on
the fundamentals of engineering electromagnetics.

About the Author

Nannapaneni Narayana Rao was born in Kakumanu, Guntur District, Andhra Pradesh, India. Prior to going to the United States in 1958, he attended high schools in Pedanandipadu and Nidubrolu; the Presidency College, Madras (now known as Chennai); and the Madras Institute of Technology. He completed high school from Nidubrolu in 1947, received the BSc. Degree in Physics from the University of Madras in 1952, and the Diploma in Electronics from the Madras Institute of Technology in 1955. In the United States, he attended the University of Washington, receiving the MS and PhD degrees in Electrical Engineering in 1960 and 1965, respectively. In 1965, he joined the faculty of the Department of Electrical Engineering, now the Department of Electrical and Computer Engineering, of the University of Illinois at Urbana-Champaign, Illinois, serving as Associate Head of the Department from 1987 to 2006.

At the University of Illinois at Urbana-Champaign, he carried out research in the general area of radiowave propagation in the ionosphere and authored the undergraduate textbooks, *Basic Electromagnetics with Applications* (Prentice Hall, 1972), and six editions of *Elements of Engineering Electromagnetics* (Prentice Hall, 1977, 1987, 1991, 1994, 2000, and 2004). The fifth edition of *Elements of Engineering Electromagnetics* was translated into Bahasa Indonesia, the language of Indonesia, by a professor of physics at the Bandung Institute of Technology, Bandung.

Professor Rao has received numerous awards and honors for his teaching and curricular activities, and contributions to engineering education in India, the USA, and Indonesia. He received the IEEE Technical Field Award in Undergraduate Teaching in

1994 with the citation, “For inspirational teaching of undergraduate students and the development of innovative instructional materials for teaching courses in electromagnetics.”

In Fall 2003, Professor Rao was named to be the first recipient of the Edward C. Jordan Professorship in Electrical and Computer Engineering, created to honor the memory of Professor Jordan, who served as department head for 25 years, and to be held by a “member of the faculty of the department who has demonstrated the qualities of Professor Jordan and whose work would best honor the legacy of Professor Jordan.”

In October 2006, Professor Rao was named the first “Distinguished Amrita Professor of Engineering” by Amrita Vishwa Vidyapeetham, Ettimadai, Coimbatore, Tamil Nadu, India, where, in July – August, 2006, he taught the inaugural course on the Indo-US Inter-University Collaborative Initiative on the Amrita e-learning network. A special Indian Edition of “Elements of Engineering Electromagnetics, Sixth Edition,” was published prior to this course offering.

Preface

"… I am talking about the areas of science and learning that have been at the heart of what we know and what we do, that which has supported and guided us and which is fundamental to our thinking. It is electromagnetism (EM) in all its many forms that has been so basic, that haunts us and guides us…" – Nick Holonyak, Jr., John Bardeen Endowed Chair Professor of Electrical and Computer Engineering and of Physics, University of Illinois at Urbana-Champaign, and the inventor of the semiconductor visible LED, laser, and quantum-well laser, in a Foreword in the Indian Edition of "Elements of Engineering Electromagnetics, Sixth Edition," by the author.

"The electromagnetic theory, as we know it, is surely one of the supreme accomplishments of the human intellect, reason enough to study it. But its usefulness in science and engineering makes it an indispensable tool in virtually any area of technology or physical research." – George W. Swenson, Jr., Professor Emeritus of Electrical and Computer Engineering, University of Illinois at Urbana-Champaign, and a pioneer in the field of radio astronomy, in the "Why Study Electromagnetics? Section in the Indian Edition of "Elements of Engineering Electromagnetics, Sixth Edition," by the author.

In this presentation, I present in a nutshell the fundamental aspects of engineering electromagnetics from the view of looking back in a reflective fashion what has already been learnt in undergraduate electromagnetics courses as a novice. The first question that comes to mind in this context is: What constitutes the fundamentals of engineering electromagnetics? If the question is posed to several individuals, it is certain that they will come up with sets of topics, not necessarily the same or in the same order, but all containing the topic, "Maxwell's Equations," at some point in the list, ranging from the beginning to the end of the list. In most cases, the response is bound to depend on the manner in which the individual was first exposed to the subject. Judging from the contents of the vast collection of undergraduate textbooks on electromagnetics, there is definitely a heavy tilt toward the traditional, or historical, approach of beginning with statics and culminating in Maxwell's equations, with perhaps an introduction to waves.

Primarily to provide a more rewarding understanding and appreciation of the subject matter, and secondarily owing to my own fascination resulting from my own experience as a student, a teacher, and an author over a few decades, I have employed here the approach of beginning with Maxwell's equations and treating the different categories of fields as solutions to Maxwell's equations. In doing so, instead of presenting the topics in an unconnected manner, I have used the thread of statics-quasistatics-waves to cover the fundamentals and bring out the frequency behavior of physical structures at the same time.

The material in this presentation is based on Chapter 1, entitled, "Fundamentals of Engineering Electromagnetics Revisited," by the author, in *Handbook of Engineering Electromagnetic*s, Marcel Dekker, 2004. I wish to express my appreciation to the following staff members of the Department of Electrical and Computer Engineering at the University of Illinois at Urbana-Champaign: Teresa Peterson for preparing the PowerPoint slides, and Kelly Collier for typing the text material.

N. Narayana Rao

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Note that this publication does not have page numbers**.** The right side pages contain slides numbered 1 through 108 in sequence. The left side pages contain explanations for the slides.

Slide Nos. 1-3

Electromagnetics is the subject having to do with electromagnetic fields. An electromagnetic field is made up of interdependent electric and magnetic fields, which is the case when the fields are varying with time, that is, they are dynamic. An electric field is a force field that acts upon material bodies by virtue of their property of charge, just as a gravitational field is a force field that acts upon them by virtue of their property of mass. A magnetic field is a force field that acts upon charges in motion.

The subject of electromagnetics is traditionally taught using the historical approach of beginning with static fields and culminating in Maxwell's equations, with perhaps an introduction to waves. Here, we employ the approach of beginning with Maxwell's equations and treating the different categories of fields as solutions to Maxwell's equations. In doing so, the thread of statics-quasistatics-waves is used to cover the fundamentals and bring out the frequency behavior of physical structures at the same time.

What is Electromagnetics?

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The Approach Used Here

The subject of electromagnetics is traditionally taught using the historical approach of beginning with static fields and culminating in Maxwell's equations, with perhaps an introduction to waves.

Here, we employ the approach of beginning with Maxwell's equations and treating the different categories of fields as solutions to Maxwell's equations. In doing so, the thread of statics-quasistatics-waves is used to cover the fundamentals and bring out the frequency behavior of physical structures at the same time.

Maxwell's Equations

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$

Electric field intensity (V/m)

Magnetic flux density $(\mathrm{Wb/m^2})$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dv$$

Charge density $(\mathrm{C/m^3})$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D} \cdot d\mathbf{S}$$

Magnetic field intensity (A/m)

Current density $(\mathrm{A/m^2})$

Displacement flux density $(\mathrm{C/m^2})$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

Slide No. 4

A region is said to be characterized by an electric field if a particle of charge q moving with a velocity $\mathbf{v}$ experiences a force $\mathbf{F}_e$, independent of $\mathbf{v}$. The force, $\mathbf{F}_e$, is given by

$$\mathbf{F}_e = q\mathbf{E} \tag{1}$$

where $\mathbf{E}$ is the electric field intensity. We note that the units of $\mathbf{E}$ are newtons per coulomb (N/C). Alternate and more commonly used units are volts per meter (V/m), where a volt is a newton-meter per coulomb. The line integral of $\mathbf{E}$ between two points A and B in an electric field region, $\int_A^B \mathbf{E} \bullet d\mathbf{l}$, has the meaning of voltage between A and B. It is the work per unit charge done by the field in the movement of the charge from A to B. The line integral of $\mathbf{E}$ around a closed path C is also known as the electromotive force or emf around C.

Electric Force on a Charge

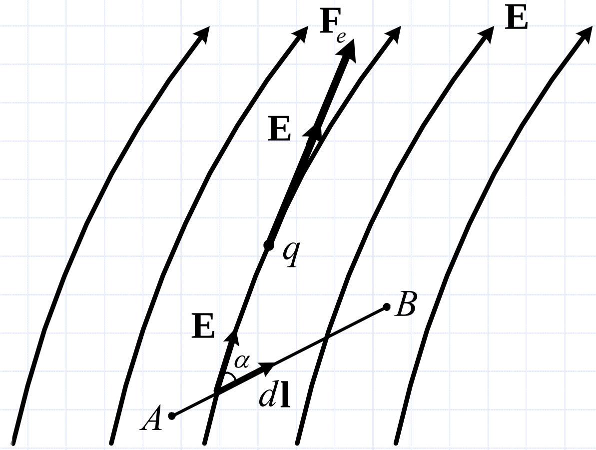
$$\mathbf{F}_e = q\,\mathbf{E}$$

N C V/m

V = (N-m)/C

$$\int_A^B \mathbf{E}\cdot d\mathbf{l} = \int_A^B (\mathbf{E}\cos\alpha)\,dl$$

= Voltage between A and B



Slide No. 5

If the charged particle experiences a force which depends on **v**, then the region is said to be characterized by a magnetic field. The force, $\mathbf{F}_m$, is given by

$$\mathbf{F}_m = q\mathbf{v} \times \mathbf{B} \qquad (2)$$

where **B** is the magnetic flux density. We note that the units of **B** are newtons/(coulomb-meter per second), or, (newton-meter per coulomb) × (seconds per square meter), or, volt-seconds per square meter. Alternate and more commonly used units are webers per square meter (Wb/m^2) or tesla (T), where a weber is a volt-second. The surface integral of **B** over a surface *S*, $\int_S \mathbf{B} \bullet d\mathbf{S}$, is the magnetic flux (Wb) crossing the surface.

Equation (2) tells us that the magnetic force is proportional to the magnitude of **v** and orthogonal to both **v** and **B** in the right-hand sense. The magnitude of the force is $qvB \sin \delta$, where δ is the angle between **v** and **B**. Since the force is normal to **v**, there is no acceleration along the direction of motion. Thus the magnetic field changes only the direction of motion of the charge, and does not alter the kinetic energy associated with it.

Magnetic Force on a Moving Charge

$$\mathbf{F}_m = q\,\mathbf{v} \times \mathbf{B}$$

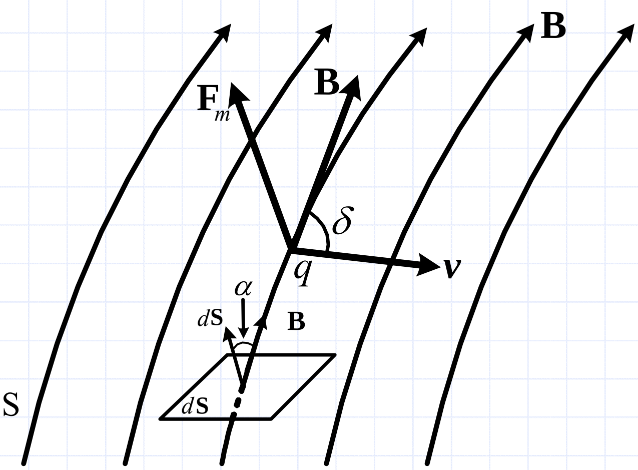
$\mathbf{F}_m$ → N, q → C, $\mathbf{v}$ → m/s, $\mathbf{B}$ → Wb/m^2

Wb = V-s

$$F_m = qvB \sin \delta$$

$$\int_S \mathbf{B} \cdot d\mathbf{S}$$

= Magnetic flux crossing S



Slide No. 6

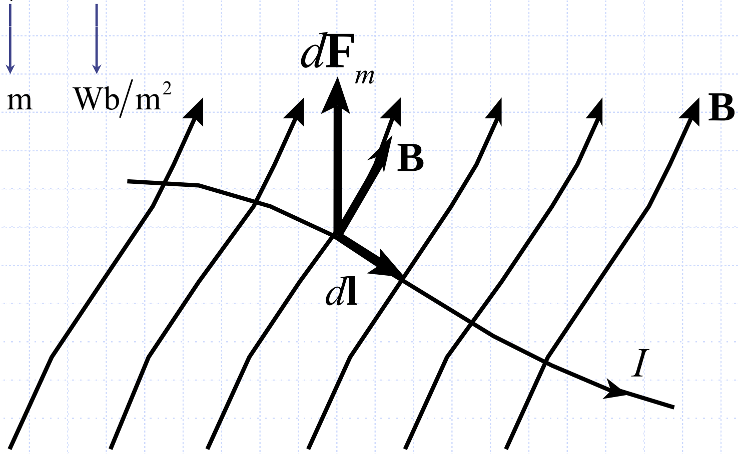
Since current flow in a wire results from motion of charges in the wire, a wire of current placed in a magnetic field experiences a magnetic force. For a differential length $d\mathbf{l}$ of a wire of current I placed in a magnetic field $\mathbf{B}$, this force is given by

$$d\mathbf{F}_{\mathrm{m}} = I\, d\mathbf{l} \times \mathbf{B} \tag{3}$$

Magnetic Force on a Current Element

$$d\mathbf{F}_m = I\, \underbrace{d\mathbf{l}} \times \mathbf{B}$$

N A m Wb/m^2



Slide No. 7

Combining (1) and (2), we obtain the expression for the total force $\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m$, experienced by a particle of charge q moving with a velocity $\mathbf{v}$ in a region of electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, respectively, as

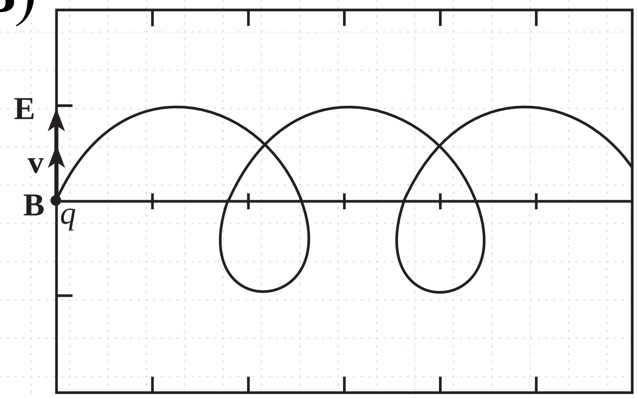
$$\begin{aligned} \mathbf{F} &= q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \\ &= q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \end{aligned} \tag{4}$$

Equation (4) is known as the Lorentz force equation.

Lorentz Force Equation

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$$

$$= q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$



Slide No. 8

The vectors **E** and **B** are the fundamental field vectors which define the force acting on a charge moving in an electromagnetic field, as given by the Lorentz force equation (4). Two associated field vectors **D** and **H**, known as the electric flux density (or the displacement flux density) and the magnetic field intensity, respectively, take into account the dielectric and magnetic properties, respectively, of material media. Materials contain charged particles which, under the application of external fields, respond giving rise to three basic phenomena known as conduction, polarization, and magnetization. Although a material may exhibit all three properties, it is classified as a conductor, a dielectric, or a magnetic material, depending upon whether conduction, polarization, or magnetization is the predominant phenomenon. While these phenomena occur on the atomic or "microscopic" scale, it is sufficient for our purpose to characterize the material based on "macroscopic" scale observations, that is, observations averaged over volumes large compared with atomic dimensions.

Materials

Materials contain charged particles that under the application of external fields respond giving rise to three basic phenomena known as *conduction*, *polarization*, and *magnetization*. While these phenomena occur on the atomic or "microscopic" scale, it is sufficient for our purpose to characterize the material based on "macroscopic" scale observations, that is, observations averaged over volumes large compared with atomic dimensions.

Slide No. 9

In the case of conductors, the effect of conduction is to produce a current in the material known as the conduction current. Conduction is the phenomenon whereby the free electrons inside the material move under the influence of the externally applied electric field with an average velocity proportional in magnitude to the applied electric field, instead of accelerating, due to the frictional mechanism provided by collisions with the atomic lattice. For linear isotropic conductors, the conduction current density, having the units of amperes per square meter (A/m^2) is related to the electric field intensity in the manner

$$\mathbf{J}_c = \sigma\mathbf{E} \tag{5}$$

where σ is the conductivity of the material, having the units siemens per meter (S/m). In semiconductors, the conductivity is governed by not only electrons but also holes.

Conductors (conduction)

$\mathbf{J}_c = \sigma \mathbf{E}$

$\mathbf{J}_c$ = Conduction current density, $\mathrm{A/m^2}$

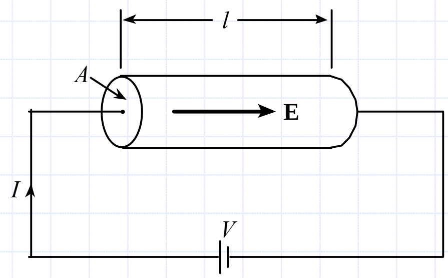
σ = Conductivity, $\mathrm{S/m}$

Ohm's Law in Circuit Theory

$V = El$

$J_c = \sigma E = \dfrac{\sigma V}{l}$

$I = J_c A = \dfrac{\sigma A}{l} V$



$V = I\left(\dfrac{l}{\sigma A}\right) = IR$ $\quad R = \dfrac{l}{\sigma A}$, resistance (ohms)

Slide Nos. 10-11

While the effect of conduction is taken into account explicitly in the electromagnetic field equations through (5), the effect of polarization is taken into account implicitly through the relationship between **D** and **E**, which is given by

$$\mathbf{D} = \varepsilon \mathbf{E} \tag{6}$$

for linear isotropic dielectrics, where ε is the permittivity of the material having the units (coulombs)^2 per (newton-meter^2), commonly known as farads per meter (F/m), where a farad is a (coulomb)^2 per newton-meter.

Polarization is the phenomenon of creation and net alignment of electric dipoles, formed by the displacements of the centroids of the electron clouds of the nuclei of the atoms within the material, along the direction of an applied electric field.

The effect of polarization is to produce a secondary field which acts in superposition with the applied field to cause the polarization. To implicitly take this into account, leading to (6), we begin with

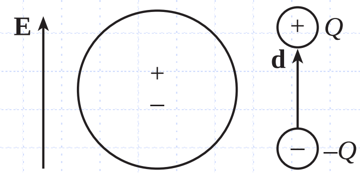
$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \tag{7}$$

where ε_0 is the permittivity of free space, having the numerical value 8.854×10^{-12}, or approximately $10^{-9}/36\pi$, and **P** is the polarization vector, or the dipole moment per unit volume, having the units (coulomb-meters) per meter^3 or coulombs per square meter. Note that this gives the units of coulombs per square meter for **D**. The term $\varepsilon_0 \mathbf{E}$ accounts for the relationship between **D** and **E** if the medium were free space, and the quantity **P** represents the effect of polarization. For linear isotropic dielectrics, **P** is proportional to **E** in the manner

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} \tag{8}$$

Dielectrics (polarization)

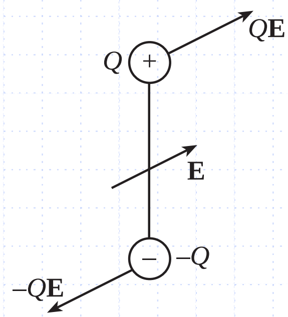
Electric Dipole



Electric Dipole Moment:

$$\mathbf{p} = Q\mathbf{d}$$

Polarization vector:



$$\mathbf{P} = \frac{1}{\Delta v}\sum_{j=1}^{N\,\Delta v}\mathbf{p}_j = N\mathbf{p}\ \left(\mathrm{C/m^2}\right)$$

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$$

ε_0 = Permittivity of free space $(\mathrm{F/m})$

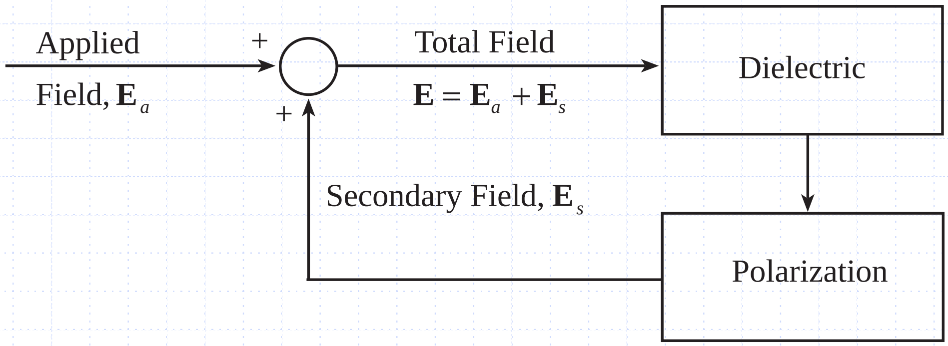
χ_e = Electric susceptibility

where χ_e, a dimensionless quantity, is the electric susceptibility, a parameter that signifies the ability of the material to get polarized. Combining (7) and (8), we have

$$\begin{aligned} \mathbf{D} &= \varepsilon_0(1 + \chi_e)\mathbf{E} \\ &= \varepsilon_0\varepsilon_r\mathbf{E} \\ &= \varepsilon\mathbf{E} \end{aligned} \tag{9}$$

where ε_r (= 1 + χ_e) is the relative permittivity of the material.

The Permittivity Concept



$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 \mathbf{E} + \varepsilon_0 \chi_e \mathbf{E}$$

$$\mathbf{D} = \varepsilon_0 (1 + \chi_e) \mathbf{E}$$

$$= \varepsilon_0 \varepsilon_r \mathbf{E} = \varepsilon \mathbf{E}, \text{ Displacement Flux Density}$$

Slide Nos. 12-13

In a similar manner, the effect of magnetization is taken into account implicitly through the relationship between **H** and **B**, which is given by

$$\mathbf{H} = \frac{\mathbf{B}}{\mu} \tag{10}$$

for linear isotropic magnetic materials, where μ is the permeability of the material, having the units newtons per (ampere)^2, commonly known as henrys per meter (H/m), where a henry is a (newton-meter) per (ampere)^2.

Magnetization is the phenomenon of net alignment of the axes of the magnetic dipoles, formed by the electron orbital and spin motion around the nuclei of the atoms in the material, along the direction of the applied magnetic field.

The effect of magnetization is to produce a secondary field which acts in superposition with the applied field to cause the magnetization. To implicitly take this into account, we begin with

$$\mathbf{B} = \mu_0\mathbf{H} + \mu_0\mathbf{M} \tag{11}$$

where μ_0 is the permeability of free space, having the numerical value $4\pi \times 10^{-7}$, and **M** is the magnetization vector or the magnetic dipole moment per unit volume, having the units (ampere-square meters) per meter^3 or amperes per meter. Note that this gives the units of amperes per meter for **H**. The term $\mu_0\mathbf{H}$ accounts for the relationship between **H** and **B** if the medium were free space, and the quantity $\mu_0\mathbf{M}$ represents the effect of magnetization. For linear isotropic magnetic materials, **M** is proportional to **H** in the manner

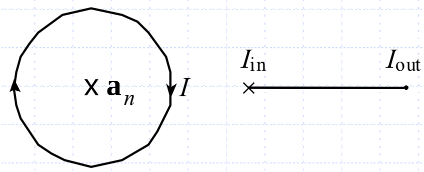
$$\mathbf{M} = \chi_m\mathbf{H} \tag{12}$$

Magnetic Materials (Magnetization)

Magnetic Dipole

Magnetic Dipole Moment:

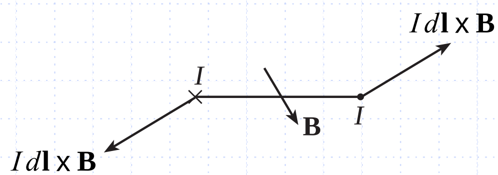
$$\mathbf{m} = I A \mathbf{a}_n$$



Magnetization Vector:

$$\mathbf{M} = \frac{1}{\Delta v} \sum_{j=1}^{N \Delta v} \mathbf{m}_j = N\mathbf{m} \, (\text{A/m})$$

$$\mathbf{M} = \frac{\chi_m}{1+\chi_m} \frac{\mathbf{B}}{\mu_0}$$



μ_0 = Permeability of free space (H/m)

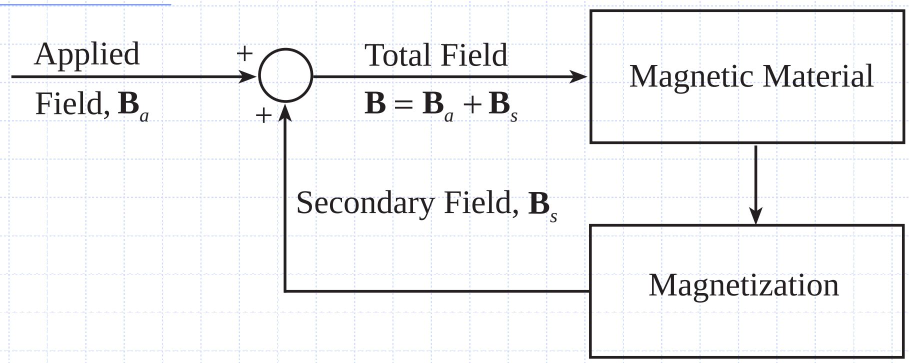
χ_m = Magnetic susceptibility

where χ_m, a dimensionless quantity, is the magnetic susceptibility, a parameter that signifies the ability of the material to get magnetized. Combining (11) and 12), we have

$$\begin{aligned}\mathbf{H} &= \frac{\mathbf{B}}{\mu_0(1+\chi_m)} \\ &= \frac{\mathbf{B}}{\mu_0\mu_r} \\ &= \frac{\mathbf{B}}{\mu}\end{aligned} \tag{13}$$

where μ_r (= 1 + χ_m) is the relative permeability of the material.

The Permeability Concept



$$\mathbf{B} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M} = \mu_0 \mathbf{H} + \mu_0 \frac{\chi_m}{1+\chi_m} \frac{\mathbf{B}}{\mu_0}$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0 (1+\chi_m)} = \frac{\mathbf{B}}{\mu_0 \mu_r} = \frac{\mathbf{B}}{\mu}$$, Magnetic Field Intensity

Slide No. 14

Equations (5), (6), and (10) are familiarly known as the constitutive relations, where σ, ε, and μ are the material parameters. The parameter σ takes into account explicitly the phenomenon of conduction, whereas the parameters ε and μ take into account implicitly the phenomena of polarization and magnetization, respectively.

Materials and Constitutive Relations

Summarizing,

$\mathbf{J}_c = \sigma \mathbf{E}$ Conductors

$\mathbf{D} = \varepsilon \mathbf{E}$ Dielectrics

$\mathbf{H} = \dfrac{\mathbf{B}}{\mu}$ Magnetic materials

E and **B** are the fundamental field vectors.

D and **H** are mixed vectors taking into account the dielectric and magnetic properties of the material implicity through ε and μ, respectively.

Slide No. 15

The constitutive relations (5), (6), and (10) tell us that $\mathbf{J}_c$ is parallel to **E**, **D** is parallel to **E**, and **H** is parallel to **B**, independent of the directions of the field vectors. For anisotropic materials, the behavior depends upon the directions of the field vectors. The constitutive relations have then to be written in matrix form. For example, in an anisotropic dielectric, each component of **P** and hence of **D** is in general dependent upon each component of **E**. Thus, in terms of components in the Cartesian coordinate system, the constitutive relation is given by

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{14}$$

or, simply by

$$[\mathbf{D}] = [\varepsilon]\,[\mathbf{E}] \tag{15}$$

where [**D**] and [**E**] are the column matrices consisting of the components of **D** and **E**, respectively, and $[\varepsilon]$ is the permittivity matrix containing the elements ε_{ij}, $i = 1, 2, 3$ and $j = 1, 2, 3$. Similar relationships hold for anisotropic conductors and anisotropic magnetic materials.

Anisotropic Materials

Example: Anisotropic Dielectrics

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

$$[\mathbf{D}] = [\varepsilon]\,[\mathbf{E}]$$

Slide No. 16

Since the permittivity matrix is symmetric, that is, $\varepsilon_{ij} = \varepsilon_{ji}$, from considerations of energy conservation, an appropriate choice of the coordinate system can be made such that some or all of the nondiagonal elements are zero. For a particular choice, all of the nondiagonal elements can be made zero so that

$$[\varepsilon] = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} \tag{16}$$

Then

$$D_{x'} = \varepsilon_1 E_{x'} \tag{17a}$$

$$D_{y'} = \varepsilon_2 E_{y'} \tag{17b}$$

$$D_{z'} = \varepsilon_3 E_{z'} \tag{17c}$$

so that **D** and **E** are parallel when they are directed along the coordinate axes, although with different values of "effective permittivity," that is, ratio of **D** to **E**, for each such direction. The axes of the coordinate system are then said to be the "principal axes" of the medium. Thus when the field is directed along a principal axis, the anisotropic medium can be treated as an isotropic medium of permittivity equal to the corresponding effective permittivity.

Specific Example of Anisotropic Dielectrics

$$[\varepsilon] = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}$$

$$D_{x'} = \varepsilon_1 E_{x'}$$

$$D_{y'} = \varepsilon_2 E_{y'}$$

$$D_{z'} = \varepsilon_3 E_{z'}$$

Characteristic Polarizations − x', y', z'

Effective Permittivities − ε_1, ε_2, ε_3

Slide Nos. 17-18

The electric and magnetic fields are governed by a set of four laws, known as the Maxwell's equations, resulting from several experimental findings and a purely mathematical contribution. Together with the constitutive relations, Maxwell's equations form the basis for the entire electromagnetic field theory. We shall consider the time variations of the fields to be arbitrary and introduce these equations and an auxiliary equation in the time domain form. In view of their experimental origin, the fundamental form of Maxwell's equations is the integral form. We shall first present all four Maxwell's equations in integral form and the auxiliary equation, the law of conservation of charge, and then discuss each equation and several points of interest pertinent to them. It is understood that all field quantities are real functions of position and time, that is, $\mathbf{E} = \mathbf{E}(\mathbf{r}, t) = \mathbf{E}(x, y, z, t)$, etc.

Maxwell's Equations in Integral Form and the Law of Conservation of Charge

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$

Faraday's Law

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D} \cdot d\mathbf{S}$$

Ampere's Circuital Law

Maxwell's Equations in Integral Form and the Law of Conservation of Charge

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dv$$

Gauss' Law for the Electric Field

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

Gauss' Law for the Magnetic Field

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{d}{dt} \int_V \rho \, dv$$

Law of Conservation of Charge

$$\oint_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt} \int_V \rho \, dv = 0$$

Slide No. 19

<u>Faraday's Law</u>

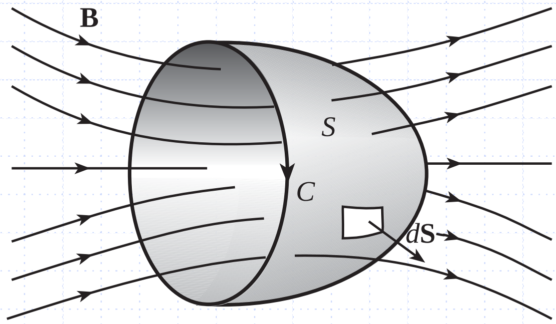
Faraday's law is a consequence of the experimental finding by Michael Faraday in 1831 that a time varying magnetic field gives rise to an electric field. Specifically, the electromotive force around a closed path *C* is equal to the negative of the time rate of increase of the magnetic flux enclosed by that path, that is,

$$\oint_C \mathbf{E} \bullet d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \bullet d\mathbf{S} \tag{18}$$

where *S* is any surface bounded by *C*.

Faraday's Law

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$



Electromotive Force (emf) or voltage around *C*
= Negative of the time rate of increase of the magnetic flux enclosed by *C*.

Slide No. 20

There are certain procedures and observations of interest pertinent to (18):

1. The direction of the infinitesimal surface vector $d\mathbf{S}$ denotes that the magnetic flux is to be evaluated in accordance with the right hand screw rule (R.H.S. rule), that is, in the sense of advance of a right hand screw as it is turned around C in the sense of C. The R.H.S. rule is a convention that is applied consistently for all electromagnetic field laws involving integration over surfaces bounded by closed paths.

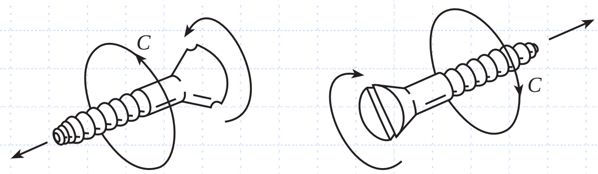
2. In evaluating the surface integrals in (18), any surface S bounded by C can be employed. This implies that the time derivative of the magnetic flux through all possible surfaces bounded by C is the same in order for the emf around C to be unique.

3. The minus sign on the right side of (18) tells us that when the magnetic flux enclosed by C is increasing with time, the induced voltage is in the sense opposite to that of C. If the path C is imagined to be occupied by a wire, then a current would flow in the wire which produces a magnetic field so as to oppose the increasing flux. Similar considerations apply for the case of the magnetic flux enclosed by C decreasing with time. These are in accordance with Lenz' law which states that the sense of the induced emf is such that any current it produces tends to oppose the change in the magnetic flux producing it.

4. If loop C contains more than one turn, such as in an N-turn coil, then the surface S bounded by C takes the shape of a spiral ramp. For a tightly wound coil, this is equivalent to the situation in which N separate, identical, single-turn loops are stacked so that the emf induced in the N-turn coil is N times the emf induced in one turn. Thus, for an N-turn coil,

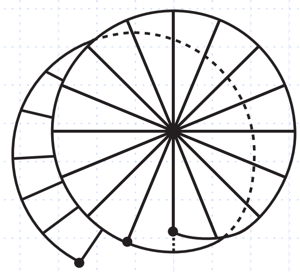
Some Considerations

RHS rule



N-turn loop

$$\text{emf} = -N\frac{d\psi}{dt}$$



ψ = Magnetic flux enclosed by one turn

$$\text{emf} = -N\frac{d\psi}{dt} \tag{19}$$

where ψ is the magnetic flux computed as though the coil is a one-turn coil.

Slide No. 21

Ampere's Circuital Law

Ampere's circuital law is a combination of an experimental finding of Oersted that electric currents generate magnetic fields and a mathematical contribution of Maxwell that time-varying electric fields give rise to magnetic fields. Specifically, the magnetomotive force, or mmf, around a closed path C is equal to the sum of the current enclosed by that path due to actual flow of charges and the displacement current due to the time rate of increase of the electric flux (or displacement flux) enclosed by that path, that is,

$$\oint_C \mathbf{H} \bullet d\mathbf{l} = \int_S \mathbf{J} \bullet d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D} \bullet d\mathbf{S} \tag{20}$$

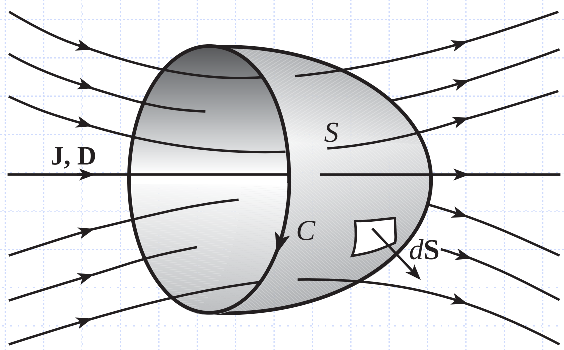
where S is any surface bounded by C.

Since magnetic force acts perpendicular to the motion of a charge, the magnetomotive force, that is, $\oint_C \mathbf{H} \bullet d\mathbf{l}$, does not have a physical meaning similar to that of the electromotive force. The terminology arises purely from analogy with electromotive force for $\oint_C \mathbf{E} \bullet d\mathbf{l}$.

In evaluating the surface integrals in (20), any surface S bounded by C can be employed. However, the same surface S must be employed for both surface integrals. This implies that the sum of the current due to flow of charges and the displacement current through all possible surfaces bounded C is the same in order for the mmf around C to be unique.

Ampere's Circuital Law

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D} \cdot d\mathbf{S}$$



Magnetomotive force (mmf) around *C*
= Current due to charges crossing *S* bounded by *C*
+ Time rate of increase of electric (or displacement) flux crossing *S*

The current density **J** in (20) pertains to true currents due to motion of true charges. It does not pertain to currents resulting from the polarization and magnetization phenomena, since these are implicitly taken into account by **D** and **H**.

Slide No. 22

The displacement current, $\frac{d}{dt}\int_S \mathbf{D} \bullet d\mathbf{S}$ is not a true current, that is, it is not a current due to actual flow of charges, such as in the case of the conduction current in wires or a convection current due to motion of a charged cloud in space. Mathematically, it has the units of $\frac{d}{dt}$ [(C/m^2) × m^2] or amperes, the same as the units for a true current, as it should be. Physically, it leads to the same phenomenon as a true current does, even in free space for which **P** is zero, and **D** is simply equal to $\varepsilon_0\mathbf{E}$. Without it, the uniqueness of the mmf around a given closed path *C* is not ensured. In fact, Ampere's circuital law in its original form did not contain the displacement current term, thereby making it valid only for the static field case. It was the mathematical contribution of Maxwell that led to the modification of the original Ampere's circuital law by the inclusion of the displacement current term. Together with Faraday's law, this modification in turn led to the theoretical prediction by Maxwell of the phenomenon of electromagnetic wave propagation in 1864 even before it was confirmed experimentally 23 years later in 1887 by Hertz.

The Displacement Current Concept

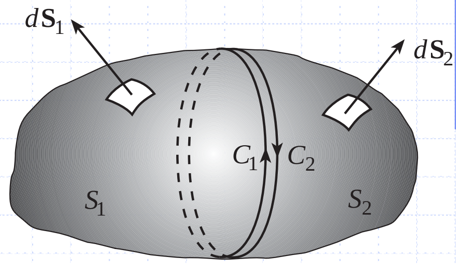
$$\oint_{C_1} \mathbf{H}\cdot d\mathbf{l} = \int_{S_1} \mathbf{J}\cdot d\mathbf{S}_1 + \frac{d}{dt}\int_{S_1} \mathbf{D}\cdot d\mathbf{S}_1$$

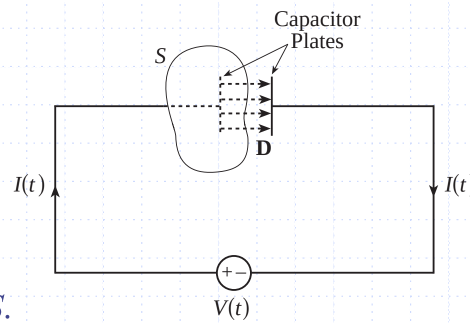
$$\oint_{C_2} \mathbf{H}\cdot d\mathbf{l} = \int_{S_2} \mathbf{J}\cdot d\mathbf{S}_2 + \frac{d}{dt}\int_{S_2} \mathbf{D}\cdot d\mathbf{S}_2$$

$$0 = \oint_{S_1+S_2} \mathbf{J}\cdot d\mathbf{S} + \frac{d}{dt}\oint_{S_1+S_2} \mathbf{D}\cdot d\mathbf{S}$$

$$\oint_S \mathbf{J}\cdot d\mathbf{S} + \frac{d}{dt}\oint_S \mathbf{D}\cdot d\mathbf{S} = 0$$

$$\frac{d}{dt}\oint_S \mathbf{D}\cdot d\mathbf{S} = -\oint_S \mathbf{J}\cdot d\mathbf{S}$$





Displacement current out of S
= Current due to flow of charges into S.

Slide No. 23

<u>Gauss' Law for the Electric Field:</u>

Gauss' law for the electric field states that electric charges give rise to electric field. Specifically, the electric flux emanating from a closed surface S is equal to the charge enclosed by that surface, that is,

$$\oint_S \mathbf{D} \bullet d\mathbf{S} = \int_V \rho \, dv \tag{21}$$

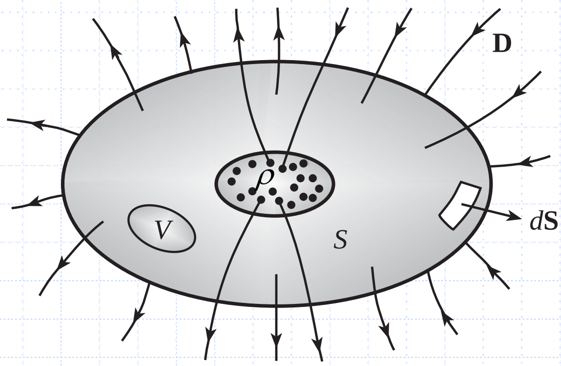
where V is the volume bounded by S. In (21), the quantity ρ is the volume charge density having the units coulombs per cubic meter (C/m^3).

The charge density ρ in (21) pertains to true charges. It does not pertain to charges resulting from the polarization phenomena, since these are implicitly taken into account by the definition of **D.**

The cut view in the figure indicates that electric field lines are discontinuous wherever there are charges, diverging from positive charges and converging on negative charges.

Gauss' Law for the Electric Field

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dv$$



Electric (or displacement) flux emanating from a closed surface *S* = Charge contained in the volume *V* bounded by *S*.

Slide No. 24

Gauss' Law for the Magnetic Field:

Gauss' law for the magnetic field states that the magnetic flux emanating from a closed surface S is equal to zero, that is,

$$\oint_S \mathbf{B} \bullet d\mathbf{S} = 0 \tag{22}$$

Thus, whatever magnetic flux enters (or leaves) a certain part of the closed surface must leave (or enter) through the remainder of the closed surface.

The observation concerning the time derivative of the magnetic flux crossing all possible surfaces bounded by a given closed path C in connection with the discussion of Faraday's law implies that the time derivative of the magnetic flux emanating from a closed surface S is zero, that is,

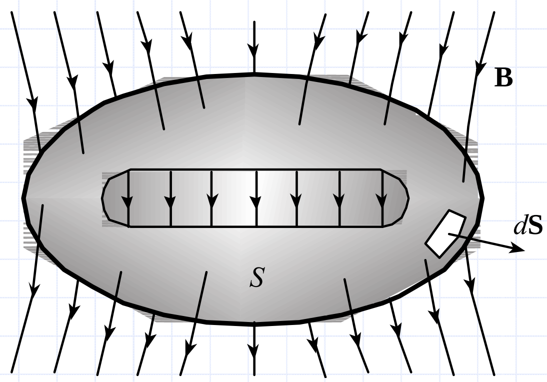
$$\frac{d}{dt}\oint_S \mathbf{B} \bullet d\mathbf{S} = 0 \tag{23}$$

One can argue then that the magnetic flux emanating from a closed surface is zero, since at an instant of time when no sources are present the magnetic field vanishes. Thus, Gauss' law for the magnetic field is not independent of Faraday's law.

The cut view in the figure indicates that magnetic field lines are continuous having no beginnings or endings.

Gauss' Law for the Magnetic Field

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$



Magnetic flux emanating from a closed surface is equal to zero.

Slide No. 25

Law of Conservation of Charge

An auxiliary equation known as the law of conservation of charge states that the current due to flow of charges emanating from a closed surface *S* is equal to the time rate of decrease of the charge inside the volume *V* bounded by that surface, that is,

$$\oint_S \mathbf{J} \bullet d\mathbf{S} = -\frac{d}{dt}\int_V \rho \, dv$$

or

$$\oint_S \mathbf{J} \bullet d\mathbf{S} + \frac{d}{dt}\int_V \rho \, dv = 0 \tag{24}$$

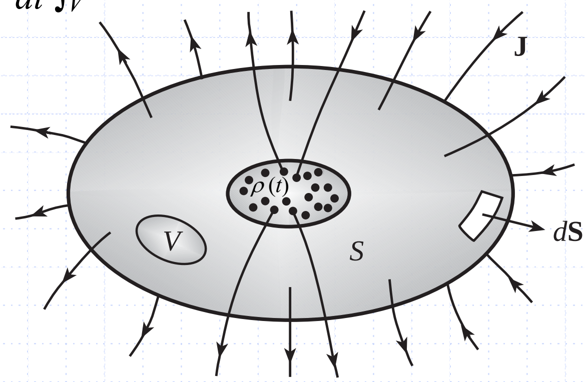
Combining the observation concerning the sum of the current due to flow of charges and the displacement current through all possible surfaces bounded by a given closed path *C* in connection with the discussion of Ampere's circuital law with the law of conservation of charge, we obtain for any closed surface *S*,

$$\frac{d}{dt}\left(\oint_S \mathbf{D} \bullet d\mathbf{S} - \int_V \rho \, dv\right) = 0 \tag{25}$$

where *V* is the volume bounded by *S*. One can then argue that the quantity inside the parentheses is zero, since at an instant of time when no sources are present, it vanishes. Thus, Gauss' law for the electric field is not independent of Ampere's circuital law in view of the law of conservation of charge.

The Law of Conservation of Charge

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{d}{dt}\int_V \rho \, dv$$



Current due to flow of charges emanating from a closed surface S
= Time rate of decrease of charge within the volume V bounded by S.

Slide Nos. 26-27

From the integral forms of Maxwell's equations, one can obtain the corresponding differential forms through the use of Stoke's and divergence theorems in vector calculus, given, respectively, by

$$\oint_C \mathbf{A} \bullet d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \bullet d\mathbf{S} \tag{26a}$$

$$\oint_S \mathbf{A} \bullet d\mathbf{S} = \int_V (\nabla \bullet \mathbf{A}) dv \tag{26b}$$

where in (26a), *S* is any surface bounded by *C*, and in (26b), *V* is the volume bounded by *S*.

Curl and Divergence

$$\nabla \times \mathbf{A} = \lim_{\Delta S \to 0} \left[\frac{\oint \mathbf{A} \cdot d\mathbf{l}}{\Delta S} \right]_{\max} \mathbf{a}_n \quad \text{Curl}$$

$$\nabla \cdot \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v} \quad \text{Divergence}$$

In Cartesian coordinates,

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \cdot \nabla \times \mathbf{A} \equiv 0 \quad \text{Identity}$$

Two Theorems

Stokes' Theorem

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$

Divergence Theorem

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_V (\nabla \cdot \mathbf{A})\, dv$$

Slide No. 28

Thus, Maxwell's equations in differential form are given by

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{27}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{28}$$

$$\nabla \bullet \mathbf{D} = \rho \tag{29}$$

$$\nabla \bullet \mathbf{B} = 0 \tag{30}$$

corresponding to the integral forms (18), (20), (21), and (22), respectively. These differential equations state that at any point in a given medium, the curl of the electric field intensity is equal to the time rate of decrease of the magnetic flux density, and the curl of the magnetic field intensity is equal to the sum of the current density due to flow of charges and the displacement current density (time derivative of the displacement flux density), whereas the divergence of the displacement flux density is equal to the volume charge density, and the divergence of the magnetic flux density is equal to zero.

Auxiliary to the Maxwell's equations in differential form is the differential equation following from the law of conservation of charge (24) through the use of (26b). Familiarly known as the continuity equation, this is given by

$$\nabla \bullet \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \tag{31}$$

It states that at any point in a given medium, the divergence of the current density due to flow of charges plus the time rate of increase of the volume charge density is equal to zero.

Maxwell's Equations in Differential Form and the Continuity Equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$ Faraday's Law

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$ Ampere's Circuital Law

$$\nabla \cdot \mathbf{D} = \rho$$ Gauss' Law for the Electric Field

$$\nabla \cdot \mathbf{B} = 0$$ Gauss' Law for the Magnetic Field

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$ Continuity Equation

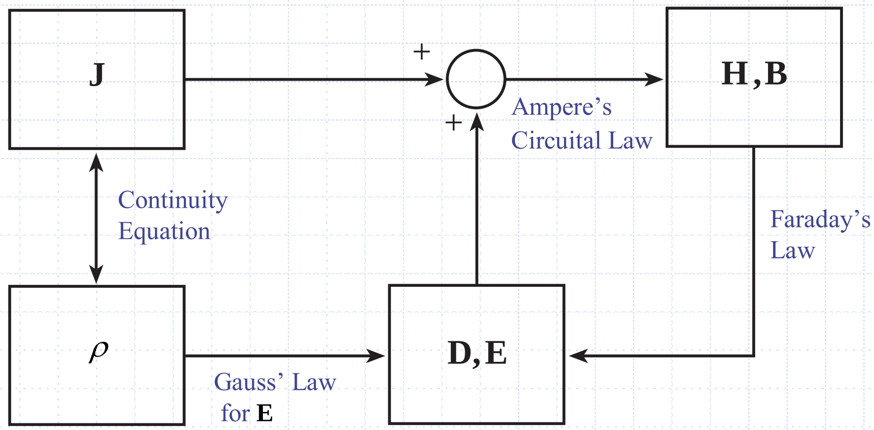
From the interdependence of the integral laws discussed in Slides 24 and 25, it follows that (30) is not independent of (27), and (29) is not independent of (28) in view of (31).

Slide No. 29

Maxwell's equations in differential form lend themselves well for a qualitative discussion of the interdependence of time-varying electric and magnetic fields giving rise to the phenomenon of electromagnetic wave propagation. Recognizing that the operations of curl and divergence involve partial derivatives with respect to space coordinates, we observe that time-varying electric and magnetic fields coexist in space, with the spatial variation of the electric field governed by the temporal variation of the magnetic field in accordance with (27), and the spatial variation of the magnetic field governed by the temporal variation of the electric field in addition to the current density in accordance with (28). Thus, if in (28) we begin with a time-varying current source represented by **J**, or a time-varying electric field represented by $\partial \mathbf{D}/dt$, or a combination of the two, then one can visualize that a magnetic field is generated in accordance with (28), which in turn generates an electric field in accordance with (27), which in turn contributes to the generation of the magnetic field in accordance with (28), and so on, as depicted on Slide 29. Note that **J** and ρ are coupled, since they must satisfy (31). Also, the magnetic field automatically satisfies (30), since (30) is not independent of (27).

The process depicted is exactly the phenomenon of electromagnetic waves propagating with a velocity (and other characteristics) determined by the parameters of the medium. In free space, the waves propagate unattenuated with the velocity $1/\sqrt{\mu_0 \varepsilon_0}$, familiarly represented by the symbol c. If either the term $\partial \mathbf{B}/\partial t$ in (27) or the

Qualitative Explanation of Electromagnetic Wave Phenomenon from Maxwell's Equations



term $\partial \mathbf{D}/\partial t$ in (28) is not present, then wave propagation would not occur. As already stated in Slide 22, it was through the addition of the term $\partial \mathbf{D}/\partial t$ in (28) that Maxwell predicted electromagnetic wave propagation before it was confirmed experimentally.

Slide Nos. 30-31

Of particular importance is the case of time variations of the fields in the sinusoidal steady state, that is, the frequency domain case. In this connection, the frequency domain forms of Maxwell's equations are of interest. Using the phasor notation based on

$$A \cos(\omega t + \phi) = \text{Re}\left[A e^{j\phi} e^{j\omega t}\right] = \text{Re}\left[\overline{A} e^{j\omega t}\right] \tag{32}$$

where $\overline{A} = Ae^{j\phi}$ is the phasor corresponding to the time function, we obtain these equations by replacing all field quantities in the time domain form of the equations by the corresponding phasor quantities and $\partial/\partial t$ by $j\omega$. Thus with the understanding that all phasor field quantities are functions of space coordinates, that is, $\overline{\mathbf{E}} = \overline{\mathbf{E}}(\mathbf{r})$, etc., we write the Maxwell's equations in frequency domain as

$$\nabla \times \overline{\mathbf{E}} = -j\omega \overline{\mathbf{B}} \tag{33}$$

$$\nabla \times \overline{\mathbf{H}} = \overline{\mathbf{J}} + j\omega \overline{\mathbf{D}} \tag{34}$$

$$\nabla \bullet \overline{\mathbf{D}} = \overline{\rho} \tag{35}$$

$$\nabla \bullet \overline{\mathbf{B}} = 0 \tag{36}$$

Also, the continuity equation (31) transforms to the frequency domain form

$$\nabla \bullet \overline{\mathbf{J}} + j\omega \overline{\rho} = 0 \tag{37}$$

Note that since $\nabla \bullet \nabla \times \overline{\mathbf{E}} = 0$, (36) follows from (33) and since $\nabla \bullet \nabla \times \overline{\mathbf{H}} = 0$, (35) follows from (34) with the aid of (37).

Sinusoidal Case

$$A\cos(\omega t+\phi)=\text{Re}\left[Ae^{j\phi}e^{j\omega t}\right]=\text{Re}\left[\overline{A}\,e^{j\omega t}\right]$$

Maxwell's Equations

$$\nabla \times \overline{\mathbf{E}} = -j\omega\overline{\mathbf{B}} \qquad \nabla\cdot\overline{\mathbf{D}}=\overline{\rho}$$

$$\nabla \times \overline{\mathbf{H}} = \overline{\mathbf{J}} + j\omega\overline{\mathbf{D}} \qquad \nabla\cdot\overline{\mathbf{B}}=0$$

Now the constitutive relations in phasor form are

$$\overline{\mathbf{D}} = \varepsilon \overline{\mathbf{E}} \tag{38a}$$

$$\overline{\mathbf{H}} = \frac{\overline{\mathbf{B}}}{\mu} \tag{38b}$$

$$\overline{\mathbf{J}}_c = \sigma \overline{\mathbf{E}} \tag{38c}$$

Sinusoidal Case

Continuity Equation

$$\nabla \cdot \bar{\mathbf{J}} + j\omega\bar{\rho} = 0$$

Constitutive Relations

$$\bar{\mathbf{D}} = \varepsilon\bar{\mathbf{E}} \qquad \bar{\mathbf{H}} = \frac{\bar{\mathbf{B}}}{\mu} \qquad \bar{\mathbf{J}}_c = \sigma\bar{\mathbf{E}}$$

Slide No. 32

Substituting (38a)-(38c) into (33)-(36), we obtain for a material medium characterized by the parameters ε, μ, and σ,

$$\nabla \times \overline{\mathbf{E}} = -j\omega\mu\overline{\mathbf{H}} \tag{39}$$

$$\nabla \times \overline{\mathbf{H}} = (\sigma + j\omega\varepsilon)\overline{\mathbf{E}} \tag{40}$$

$$\nabla \bullet \overline{\mathbf{H}} = 0 \tag{41}$$

$$\nabla \bullet \overline{\mathbf{E}} = \frac{\overline{\rho}}{\varepsilon} \tag{42}$$

Note however that if the medium is homogeneous, that is, if the material parameters are independent of the space coordinates, (40) gives

$$\nabla \bullet \overline{\mathbf{E}} = \frac{1}{\sigma + j\omega\varepsilon} \nabla \bullet \nabla \times \overline{\mathbf{H}} = 0 \tag{43}$$

so that $\overline{\rho} = 0$ in such a medium.

A point of importance in connection with the frequency domain form of Maxwell's equations is that in these equations, the parameters ε, μ, and σ can be allowed to be functions of ω. In fact, for many dielectrics, the conductivity increases with frequency in such a manner that the quantity $\sigma/\omega\varepsilon$ is more constant than is the conductivity. This quantity is the ratio of the magnitudes of the two terms on the right side of (40), that is, the conduction current density term $\sigma\overline{\mathbf{E}}$ and the displacement current density term $j\omega\varepsilon\overline{\mathbf{E}}$.

Maxwell's Equations for the Sinusoidal Case for a Material Medium

$$\nabla \times \overline{\mathbf{E}} = -j\omega\mu\overline{\mathbf{H}}$$

$$\nabla \cdot \overline{\mathbf{H}} = 0$$

$$\nabla \times \overline{\mathbf{H}} = (\sigma + j\omega\varepsilon)\overline{\mathbf{E}}$$

$$\nabla \cdot \overline{\mathbf{E}} = \frac{\overline{\rho}}{\varepsilon}$$

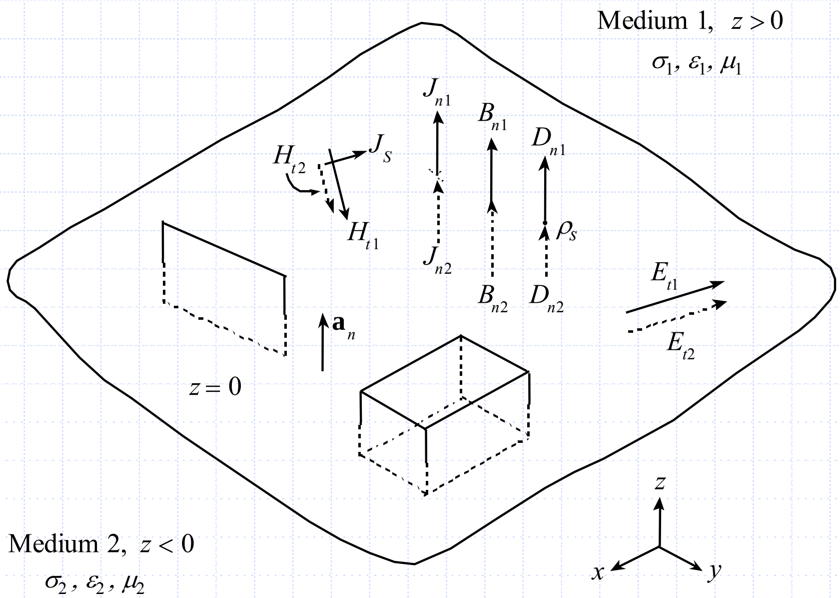
$$\nabla \cdot \overline{\mathbf{E}} = \frac{1}{\sigma + j\omega\varepsilon}\nabla \cdot \nabla \times \overline{\mathbf{H}} = 0$$

Slide Nos. 33-35

Maxwell's equations in differential form govern the interrelationships between the field vectors and the associated source densities at points in a given medium. For a problem involving two or more different media, the differential equations pertaining to each medium provide solutions for the fields that satisfy the characteristics of that medium. These solutions need to be matched at the boundaries between the media by employing "boundary conditions," which relate the field components at points adjacent to and on one side of a boundary to the field components at points adjacent to and on the other side of that boundary. The boundary conditions arise from the fact that the integral equations involve closed paths and surfaces and they must be satisfied for all possible closed paths and surfaces whether they lie entirely in one medium or encompass a portion of the boundary.

The boundary conditions are obtained by considering one integral equation at a time and applying it to a closed path or a closed surface encompassing the boundary, and in the limit that the area enclosed by the closed path, or the volume bounded by the closed surface goes to zero. Let the quantities pertinent to medium 1 be denoted by subscript 1 and the quantities pertinent to medium 2 be denoted by subscript 2, $\mathbf{a}_n$ be the unit normal vector to the surface and directed into medium 1. Let all normal components at the boundary in both media be directed along $\mathbf{a}_n$ and denoted by an additional subscript n, and all tangential components at the boundary in both media be denoted by an additional subscript t. Let the surface charge density (C/m^2) and the surface current density (A/m) on the boundary be ρ_S and $\mathbf{J}_S$, respectively. Then, the boundary conditions corresponding to the Maxwell's equations in integral form can be summarized as

Boundary Conditions



$$\mathbf{a}_n \times (\mathbf{E}_1 - \mathbf{E}_2) = \mathbf{0} \tag{44a}$$

$$\mathbf{a}_n \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_S \tag{44b}$$

$$\mathbf{a}_n \bullet (\mathbf{D}_1 - \mathbf{D}_2) = \rho_S \tag{44c}$$

$$\mathbf{a}_n \bullet (\mathbf{B}_1 - \mathbf{B}_2) = 0 \tag{44d}$$

or in scalar form,

$$E_{t1} - E_{t2} = 0 \tag{45a}$$

$$H_{t1} - H_{t2} = J_S \tag{45b}$$

$$D_{n1} - D_{n2} = \rho_S \tag{45c}$$

$$B_{n1} - B_{n2} = 0 \tag{45d}$$

In words, the boundary conditions state that at a point on the boundary, the tangential components of **E** and the normal components of **B** are continuous, whereas the tangential components of **H** are discontinuous by the amount equal to J_S at that point, and the normal components of **D** are discontinuous by the amount equal to ρ_S at that point, as illustrated on Slide 33. It should be noted that the information concerning the direction of $\mathbf{J}_S$ relative to that of $(\mathbf{H}_1 - \mathbf{H}_2)$, which is contained in (44b), is not present in (45b). Hence, in general, (45b) is not sufficient and it is necessary to use (44b).

While (44a)-(44d) or (45a)-(45d) are the most commonly used boundary conditions, another useful boundary condition resulting from the law of conservation of charge is given by

$$\mathbf{a}_n \bullet (\mathbf{J}_1 - \mathbf{J}_2) = -\nabla_S \bullet \mathbf{J}_S - \frac{\partial \rho_S}{\partial t} \tag{46}$$

In words, (46) states that, at any point on the boundary, the components of $\mathbf{J}_1$ and $\mathbf{J}_2$ normal to the boundary are discontinuous by the amount equal to the negative of the sum

Boundary Conditions

$$\mathbf{a}_n \times (\mathbf{E}_1 - \mathbf{E}_2) = \mathbf{0}$$

from Faraday's Law

$$\mathbf{a}_n \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_S$$

from Ampere's Circuital Law

$$\mathbf{a}_n \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_S$$

from Gauss' Law for the Electric Field

$$\mathbf{a}_n \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$$

from Gauss' Law for the Magnetic Field

$$\mathbf{a}_n \cdot (\mathbf{J}_1 - \mathbf{J}_2) = -\nabla_S \cdot \mathbf{J}_S - \frac{\partial \rho_S}{\partial t}$$

from the Law of Conservation of Charge

of the two-dimensional divergence of the surface current density and the time derivative of the surface charge density at that point.

Boundary Conditions in Scalar Form

$E_{t1} - E_{t2} = 0$	Tangential component of **E** is continuous.
$H_{t1} - H_{t2} = J_S$	Tangential component of **H** is discontinuous by the amount equal to the surface current density.
$D_{n1} - D_{n2} = \rho_S$	Normal component of **D** is discontinuous by the amount equal to the surface charge density.
$B_{n1} - B_{n2} = 0$	Normal component of **B** is continuous.

Slide Nos. 36-40

Maxwell's equations in differential form, together with the constitutive relations and boundary conditions, allow for the unique determination of the fields **E**, **B**, **D**, and **H**, for a given set of source distributions with densities **J** and ρ. An alternate approach involving the electric scalar potential, Φ, and the magnetic vector potential, **A**, known together as the electromagnetic potentials from which the fields can be derived, simplifies the solution in some cases. This approach leads to solving two separate differential equations, one for Φ involving ρ alone, and the second for **A**, involving **J** alone.

To obtain these equations, we first note that in view of (30), **B** can be expressed as the curl of another vector. Thus

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{47}$$

Note that the units of **A** are the units of **B** times meter, that is, Wb/m. Now, substituting (47) into (27), interchanging the operations of $\partial/\partial t$ and curl, and rearranging, we obtain

$$\nabla \times \left[\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right] = 0$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla\Phi$$

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} \tag{48}$$

where the negative sign associated with $\nabla\Phi$ is chosen for a reason to be evident later in Slide 47. Note that the units of Φ are the units of **E** times meter, that is, V. Note also that the knowledge of Φ and **A** enables the determination of **E** and **B**, from which **D** and **H** can be found by using the constitutive relations.

Now, using (6) and (10) to obtain **D** and **H** in terms of Φ and **A**, and substituting into (29) and (28), we obtain

Electromagnetic Potentials

$$\nabla \cdot \mathbf{B} = 0 \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \left[\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right] = 0$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi \qquad \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$$

Φ = Electric scalar potential

$\mathbf{A}$ = Magnetic vector potential

$$\nabla^2\Phi + \nabla \bullet \left[\frac{\partial \mathbf{A}}{\partial t}\right] = -\frac{\rho}{\varepsilon} \tag{49a}$$

$$\nabla \times \nabla \times \mathbf{A} + \mu\varepsilon \frac{\partial}{\partial t}\left[\nabla\Phi + \frac{\partial \mathbf{A}}{\partial t}\right] = \mu\mathbf{J} \tag{49b}$$

where we have assumed the medium to be homogeneous and isotropic, in addition to being linear. Using the vector identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \bullet \mathbf{A}) - \nabla^2\mathbf{A} \tag{50}$$

and interchanging the operations of $\frac{\partial}{\partial t}$ and divergence or gradient depending on the term, and rearranging, we get

$$\nabla^2\Phi + \frac{\partial}{\partial t}(\nabla \bullet \mathbf{A}) = -\frac{\rho}{\varepsilon} \tag{51a}$$

$$\nabla^2\mathbf{A} - \nabla\left[\nabla \bullet \mathbf{A} + \mu\varepsilon\frac{\partial \Phi}{\partial t}\right] - \mu\varepsilon\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu\mathbf{J} \tag{51b}$$

These equations are coupled. To uncouple them, we make use of Helmholtz's theorem, which states that a vector field is completely specified by its curl and divergence. Therefore, since the curl of **A** is given by (47), we are at liberty to specify the divergence of **A**. We do this by setting

$$\nabla \bullet \mathbf{A} = -\mu\varepsilon\frac{\partial \Phi}{\partial t} \tag{52}$$

which is known as the Lorenz condition, resulting in the uncoupled equations

$$\nabla^2\Phi - \mu\varepsilon\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\varepsilon} \tag{53}$$

$$\nabla^2\mathbf{A} - \mu\varepsilon\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu\mathbf{J} \tag{54}$$

which are called the potential function equations. While the Lorenz condition may appear to be arbitrary, it actually implies the continuity equation, which can be shown by taking the Laplacian on both sides of (52) and using (53) and (54).

Gradient and Laplacian

$$\nabla\Phi = \frac{\partial\Phi}{\partial n}\,\mathbf{a}_n$$ Gradient

In Cartesian coordinates,

$$\nabla\Phi = \frac{\partial\Phi}{\partial x}\,\mathbf{a}_x + \frac{\partial\Phi}{\partial y}\,\mathbf{a}_y + \frac{\partial\Phi}{\partial z}\,\mathbf{a}_z$$

$$\nabla \times \nabla\Phi \equiv 0$$ Identity

$$\nabla \cdot \nabla\Phi = \nabla^2\Phi$$ Laplacian of a scalar

In Cartesian coordinates,

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$$

It can be seen that (53) and (54) are not only uncoupled but they are also similar, particularly in Cartesian coordinates, since (54) decomposes into three equations involving the three Cartesian components of **J**, each of which is similar to (53). By solving (53) and (54), one can obtain the solutions for Φ and **A**, respectively, from which **E** and **B** can be found by using (48) and (47), respectively. In practice, however, since ρ is related to **J** through the continuity equation, it is sufficient to find **B** from **A** obtained from the solution of (54) and then find **E** by using the Maxwell's equation for the curl of **H**, given by (28).

Potential Function Equations

$$\nabla \cdot \mathbf{D} = \rho \qquad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla^2 \Phi + \nabla \cdot \left(\frac{\partial \mathbf{A}}{\partial t} \right) = -\frac{\rho}{\varepsilon}$$

$$\nabla \times \nabla \times \mathbf{A} + \mu\varepsilon \frac{\partial}{\partial t}\left(\nabla \Phi + \frac{\partial \mathbf{A}}{\partial t} \right) = \mu \mathbf{J}$$

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$ Identity

Laplacian of a Vector

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$$

In Cartesian coordinates

$$\nabla^2 \mathbf{A} = \nabla\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) - \nabla \times \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= (\nabla^2 A_x)\mathbf{a}_x + (\nabla^2 A_y)\mathbf{a}_y + (\nabla^2 A_z)\mathbf{a}_z$$

Potential Function Equations

$$\nabla^2\Phi + \frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) = -\frac{\rho}{\varepsilon}$$

$$\nabla^2\mathbf{A} - \nabla\left(\nabla\cdot\mathbf{A} + \mu\varepsilon\frac{\partial\Phi}{\partial t}\right) - \mu\varepsilon\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu\mathbf{J}$$

$$\nabla\cdot\mathbf{A} = -\mu\varepsilon\frac{\partial\Phi}{\partial t}$$

Lorenz Condition

$$\nabla^2\Phi - \mu\varepsilon\frac{\partial^2\Phi}{\partial t^2} = -\frac{\rho}{\varepsilon}$$

for Electric Scalar Potential

$$\nabla^2\mathbf{A} - \mu\varepsilon\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu\mathbf{J}$$

for Magnetic Vector Potential

Slide Nos. 41-43

A unique property of the electromagnetic field is its ability to transfer power between two points even in the absence of an intervening material medium. Without such ability, the effect of the field generated at one point will not be felt at another point, and hence the power generated at the first point cannot be put to use at the second point.

To discuss power flow associated with an electromagnetic field, we begin with the vector identity

$$\nabla \bullet (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \bullet (\nabla \times \mathbf{E}) - \mathbf{E} \bullet (\nabla \times \mathbf{H}) \tag{55}$$

and make use of Maxwell's curl equations (27) and (28) to write

$$\nabla \bullet (\mathbf{E} \times \mathbf{H}) = -\mathbf{E} \bullet \mathbf{J} - \mathbf{E} \bullet \frac{\partial \mathbf{D}}{\partial t} - \mathbf{H} \bullet \frac{\partial \mathbf{B}}{\partial t} \tag{56}$$

Allowing for conductivity of a material medium by denoting $\mathbf{J} = \mathbf{J}_0 + \mathbf{J}_c$, where $\mathbf{J}_0$ is that part of $\mathbf{J}$ that can be attributed to a source, and using the constitutive relations (5), (6), and (10), we obtain for a medium characterized by σ, ε, and μ,

$$-\mathbf{E} \bullet \mathbf{J}_0 = \sigma\varepsilon^2 + \frac{\partial}{\partial t}\left[\frac{1}{2}\varepsilon E^2\right] + \frac{\partial}{\partial t}\left[\frac{1}{2}\mu H^2\right] + \nabla \bullet (\mathbf{E} \times \mathbf{H}) \tag{57}$$

Defining a vector $\mathbf{P}$ given by

$$\mathbf{P} = \mathbf{E} \times \mathbf{H} \tag{58}$$

and taking the volume integral of both sides of (58), we obtain

$$-\int_V (\mathbf{E} \bullet \mathbf{J}_0)\, dv = \int_V \sigma E^2\, dv + \frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\varepsilon E^2\right) dv + \frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\mu H^2\right) dv + \oint_S \mathbf{P} \bullet d\mathbf{S} \tag{59}$$

where we have also interchanged the differentiation operation with time and integration operation over volume in the second and third terms on the right side and used the divergence theorem for the last term.

Power Flow and Energy Storage

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H})$$

A Vector Identity

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\mathbf{E} \cdot \mathbf{J} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}$$

For $\mathbf{J} = \mathbf{J}_0 + \mathbf{J}_c = \mathbf{J}_0 + \sigma \mathbf{E},$

$$-\mathbf{E} \cdot \mathbf{J}_0 = \sigma E^2 + \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right) + \nabla \cdot (\mathbf{E} \times \mathbf{H})$$

Define $\mathbf{P} = \mathbf{E} \times \mathbf{H}$ Poynting Vector

In (59), the left side is the power supplied to the field by the current source $\mathbf{J}_0$ inside V. The quantities $\sigma E^2, \frac{1}{2}\varepsilon E^2$, and $\frac{1}{2}\mu H^2$ are the power dissipation density (W/m^3), the electric stored energy density (J/m^3), and the magnetic stored energy density (J/m^3), respectively, due to the conductive, dielectric, and magnetic properties, respectively, of the medium. Hence, (59) says that the power delivered to the volume V by the current source $\mathbf{J}_0$ is accounted for by the power dissipated in the volume due to the conduction current in the medium, plus the time rates of increase of the energies stored in the electric and magnetic fields, plus another term, which we must interpret as the power carried by the electromagnetic field out of the volume V, for conservation of energy to be satisfied. It then follows that the vector **P** has the meaning of power flow density vector associated with the electromagnetic field. The statement represented by (59) is known as the Poynting's theorem and the vector **P** is known as the Poynting vector. We note that the units of $\mathbf{E} \times \mathbf{H}$ are volts per meter times amperes per meter, or watts per square meter (W/m^2) and do indeed represent power density. In particular, since **E** and **H** are instantaneous field vectors, $\mathbf{E} \times \mathbf{H}$ represents the instantaneous Poynting vector. Note that the Poynting's theorem tells us only that the power flow out of a volume V is given by the surface integral of the Poynting vector over the surface S bounding that volume. Hence we can add to **P** any vector for which the surface integral over S vanishes, without affecting the value of the surface integral. However, generally, we are interested in the total power leaving a closed surface and the interpretation of **P** alone as representing the power flow density vector is sufficient.

Poynting's Theorem

$$-\int_V (\mathbf{E}\cdot\mathbf{J}_0)\,dv = \int_V \sigma E^2\,dv + \frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\varepsilon E^2\right) dv + \frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\mu H^2\right) dv + \oint_S \mathbf{P}\cdot d\mathbf{S}$$

Source power density

Power dissipation density

Electric stored energy density

Magnetic stored energy density

Interpretation of Poynting's Theorem

Poynting's Theorem says that the power delivered to the volume *V* by the current source $\mathbf{J}_0$ is accounted for by the power dissipated in the volume due to the conduction current in the medium, plus the time rates of increase of the energies stored in the electric and magnetic fields, plus another term, which we must interpret as the power carried by the electromagnetic field out of the volume *V*, for conservation of energy to be satisfied. It then follows that the Poynting vector **P** has the meaning of power flow density vector associated with the electromagnetic field. We note that the units of $\mathbf{E} \times \mathbf{H}$ are volts per meter times amperes per meter, or watts per square meter (W/m^2) and do indeed represent power density.

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For sinusoidally time-varying fields, that is, for the frequency domain case, the quantity of importance is the time average Poynting vector instead of the instantaneous Poynting vector. The time-average Poynting vector, denoted by < **P** >, is given by

$$< \mathbf{P} > = \mathrm{Re}\left[\overline{\mathbf{P}}\right] \tag{60}$$

where $\overline{\mathbf{P}}$ is the complex Poynting vector given by

$$\overline{\mathbf{P}} = \frac{1}{2}\overline{\mathbf{E}} \times \overline{\mathbf{H}}^* \tag{61}$$

where the star denotes complex conjugate. The Poynting theorem for the frequency domain case, known as the complex Poynting's theorem, is given by

$$-\int_V \left(\frac{1}{2}\overline{\mathbf{E}} \bullet \overline{\mathbf{J}}_0^*\right) dv = \int_V < p_d > dv + j2\omega \int_V \left(< w_m > - < w_e >\right) dv + \oint_S \overline{\mathbf{P}} \bullet d\mathbf{S} \tag{62}$$

where

$$< p_d > = \frac{1}{2}\sigma\overline{\mathbf{E}} \bullet \overline{\mathbf{E}}^* \tag{63a}$$

$$< w_e > = \frac{1}{4}\varepsilon\overline{\mathbf{E}} \bullet \overline{\mathbf{E}}^* \tag{63b}$$

$$< w_m > = \frac{1}{4}\mu\overline{\mathbf{H}} \bullet \overline{\mathbf{H}}^* \tag{63c}$$

are the time-average power dissipation density, the time-average electric stored energy density, and the time-average magnetic stored energy density, respectively. Equation (62) states that the time-average, or real, power delivered to the volume V by the current source is accounted for by the time-average power dissipated in the volume plus the time-average power carried by the electromagnetic field out of the volume through the surface S bounding the volume, and that the reactive power delivered to the volume V by the current source is equal to the reactive power carried by the electromagnetic field out of

Complex Poynting Vector and Complex Poynting's Theorem

$$\langle \mathbf{P} \rangle = \mathrm{Re}\left[\overline{\mathbf{P}}\right] \qquad \overline{\mathbf{P}} = \frac{1}{2}\overline{\mathbf{E}} \times \overline{\mathbf{H}}^*$$

$$-\int_V \left(\frac{1}{2}\overline{\mathbf{E}} \cdot \overline{\mathbf{J}}_0^*\right) dv = \int_V < p_d > dv + j2\omega \int_V \left(< w_m > - < w_e >\right) dv + \oint_S \overline{\mathbf{P}} \cdot d\mathbf{S}$$

$$< p_d > = \frac{1}{2}\sigma \overline{\mathbf{E}} \cdot \overline{\mathbf{E}}^* \qquad < w_e > = \frac{1}{4}\varepsilon \overline{\mathbf{E}} \cdot \overline{\mathbf{E}}^* \qquad < w_m > = \frac{1}{4}\mu \overline{\mathbf{H}} \cdot \overline{\mathbf{H}}^*$$

The time-average, or real, power delivered to the volume V by the current source is accounted for by the time-average power dissipated in the volume plus the time-average power carried by the electromagnetic field out of the volume through the surface S bounding the volume. The reactive power delivered to the volume V by the current source is equal to the reactive power carried by the electromagnetic field out of the volume V through the surface S plus a quantity that is 2ω times the difference between the time-average magnetic and electric stored energies in the volume.

the volume V through the surface S plus a quantity which is 2ω times the difference between the time-average magnetic and electric stored energies in the volume.

Slide No. 45

While every macroscopic field obeys Maxwell's equations in their entirety, depending on their most dominant properties it is sufficient to consider a subset of, or certain terms only, in the equations. The primary classification of fields is based on their time dependence. Fields which do not change with time are called *static*. Field which change with time are called *dynamic*. Static fields are the simplest kind of fields, because for them $\partial/\partial t = 0$ and all terms involving differentiation with respect to time go to zero. Dynamic fields are the most complex, since for them Maxwell's equations in their entirety must be satisfied, resulting in wave type solutions, as provided by the qualitative explanation earlier. However, if certain features of the dynamic field can be analyzed as though the field were static, then the field is called *quasistatic*. If the important features of the field are not amenable to static type field analysis, they are generally referred to as *time-varying*, although in fact, quasistatic fields are also time-varying. Since in the most general case, time-varying fields give rise to wave phenomena, involving velocity of propagation and time delay that cannot be neglected, it can be said that quasistatic fields are those time-varying fields for which wave propagation effects can be neglected.

Classification of Fields

Static Fields (No time variation; $\partial / \partial t = 0$)

Static electric, or electrostatic fields

Static magnetic, or magnetostatic fields

Electromagnetostatic fields

Dynamic Fields (Time-varying)

Quasistatic Fields (Dynamic fields that can be analyzed as though the fields are static)

Electroquasistatic fields

Magnetoquasistatic fields

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Static Fields

For static fields, $\partial/\partial t = 0$. Maxwell's equations in integral form and the law of conservation of charge become

$$\oint_C \mathbf{E} \bullet d\mathbf{l} = 0 \qquad (64a)$$

$$\oint_C \mathbf{H} \bullet d\mathbf{l} = \int_S \mathbf{J} \bullet d\mathbf{S} \qquad (64b)$$

$$\oint_S \mathbf{D} \bullet d\mathbf{S} = \int_V \rho \, dv \qquad (64c)$$

$$\oint_S \mathbf{B} \bullet d\mathbf{S} = 0 \qquad (64d)$$

$$\oint_S \mathbf{J} \bullet d\mathbf{S} = 0 \qquad (64e)$$

whereas Maxwell's equations in differential form and the continuity equation reduce to

$$\nabla \times \mathbf{E} = 0 \qquad (65a)$$

$$\nabla \times \mathbf{H} = \mathbf{J} \qquad (65b)$$

$$\nabla \bullet \mathbf{D} = \rho \qquad (65c)$$

$$\nabla \bullet \mathbf{B} = 0 \qquad (65d)$$

$$\nabla \bullet \mathbf{J} = 0 \qquad (65e)$$

Immediately, one can see that, unless **J** includes a component due to conduction current, the equations involving the electric field are completely independent of those involving the magnetic field. Thus the fields can be subdivided into *static electric fields*, or *electrostatic fields*, governed by (64a) and (64c), or (65a) and (65c), and *static magnetic fields*, or *magnetostatic fields*, governed by (64b) and (64d), or (65b) and (65d).

Static Fields

For static fields, $\partial / \partial t = 0$, and the equations reduce to

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \qquad \nabla \times \mathbf{E} = 0$$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} \qquad \nabla \times \mathbf{H} = \mathbf{J}$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho \, dv \qquad \nabla \cdot \mathbf{D} = \rho$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = 0 \qquad \nabla \cdot \mathbf{J} = 0$$

The source of a static electric field is ρ, whereas the source of a static magnetic field is **J**. One can also see from (64e) or (65e) that no relationship exists between **J** and ρ. If **J** includes a component due to conduction current, then, since $\mathbf{J}_c = \sigma\mathbf{E}$, a coupling between the electric and magnetic fields exists for that part of the total field associated with $\mathbf{J}_c$. However, the coupling is only one way, since the right side of (64a) or (65a) is still zero. The field is then referred to as *electromagnetostatic field*. It can also be seen then that for consistency, the right sides of (64c) and (65c) must be zero, since the right sides of (64e) and (65e) are zero. We shall now consider each of the three types of static fields separately and discuss some fundamental aspects.

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Electrostatic Fields

The equations of interest are (64a) and (64c), or (65a) and (65c). The first of each pair of these equations simply tells us that the electrostatic field is a conservative field, and the second of each pair of these equations enables us, in principle, to determine the electrostatic field for a given charge distribution. Alternatively, the potential function equation (53), which reduces to

$$\nabla^2\Phi = -\frac{\rho}{\varepsilon} \tag{66}$$

can be used to find the electric scalar potential, Φ, from which the electrostatic field can be determined by using (48), which reduces to

$$\mathbf{E} = -\nabla\Phi \tag{67}$$

Equation (66) is known as the Poisson’s equation, which automatically includes the condition that the field be conservative. It is worth noting that the potential difference

Electrostatic Fields

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \qquad \nabla \times \mathbf{E} = 0$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho \, dv \qquad \nabla \cdot \mathbf{D} = \rho$$

$$\nabla^2 \Phi = -\frac{\rho}{\varepsilon}$$ Poission's Equation

$$\mathbf{E} = -\nabla\Phi$$

$$\int_A^B \mathbf{E} \cdot d\mathbf{l} = \int_A^B [-\nabla\Phi] \cdot d\mathbf{l} = \Phi_A - \Phi_B$$

between two points A and B in the static electric field, which is independent of the path followed from A to B because of the conservative nature of the field is

$$\begin{aligned}\int_A^B \mathbf{E} \bullet d\mathbf{l} &= \int_A^B [-\nabla\Phi] \bullet d\mathbf{l} \\ &= \Phi_A - \Phi_B\end{aligned} \tag{68}$$

the difference between the value of Φ at A and the value of Φ at B. The choice of minus sign associated with $\nabla\Phi$ in (48) is now evident.

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The solution to Poisson's equation (66) for a given charge density distribution $\rho(\mathbf{r})$ is given by

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon} \int_{V'} \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dv' \tag{69}$$

where the prime denotes source point and no prime denotes field point. Although cast in terms of volume charge density, (69) can be formulated in terms of a surface charge distribution, a line charge distribution, or a collection of point charges. In particular, for a point charge $Q(\mathbf{r}')$, the solution is given by

$$\Phi(\mathbf{r}) = \frac{Q(\mathbf{r}')}{4\pi\varepsilon|\mathbf{r} - \mathbf{r}'|} \tag{70}$$

It follows from (67) that the electric field intensity due to the point charge is given by

$$\mathbf{E}(\mathbf{r}) = \frac{Q(\mathbf{r}')(\mathbf{r} - \mathbf{r}')}{4\pi\varepsilon|\mathbf{r} - \mathbf{r}'|^3} \tag{71}$$

which is exactly the expression that results from Coulomb's law for the electric force between two point charges.

Solution for Potential and Field

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon} \int_{V'} \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, dv'$$

Solution for charge distribution

$$\Phi(\mathbf{r}) = \frac{Q(\mathbf{r}')}{4\pi\varepsilon|\mathbf{r} - \mathbf{r}'|}$$

Solution for point charge

$$\mathbf{E}(\mathbf{r}) = \frac{Q(\mathbf{r}')(\mathbf{r} - \mathbf{r}')}{4\pi\varepsilon|\mathbf{r} - \mathbf{r}'|^3}$$

Electric field due to point charge

Equation (69) or its alternate forms can be used to solve two types of problems: (a) finding the electrostatic potential for a specified charge distribution by evaluating the integral on the right side, which is a straightforward process with the help of a computer but can be considerably difficult analytically except for a few examples, and (b) finding the surface charge distribution on the surfaces of an arrangement of conductors raised to specified potentials, by inversion of the equation, which is the basis for numerical solution by the well-known *method of moments*. In the case of (a), the electric field can then be found by using (67).

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In a charge-free region, $\rho = 0$, and Poisson's equation (66) reduces to

$$\nabla^2\Phi = 0 \tag{72}$$

which is known as the Laplace's equation. The field is then due to charges outside the region, such as surface charge on conductors bounding the region. The situation is then one of solving a boundary value problem. In general, for arbitrarily-shaped boundaries, a numerical technique, such as the *method of finite differences*, is employed for solving the problem. Here, we consider analytical solution involving one-dimensional variation of Φ.

Laplace's Equation and One-Dimensional Solution

For $\rho = 0$, Poission's equation reduces to

$\nabla^2 \Phi = 0$ Laplace's equation

$$\frac{d^2\Phi}{dx^2} = 0 \qquad\qquad \Phi = Ax + B$$

Slide Nos. 50-51

A simple example is that of the parallel-plate arrangement in which two parallel, perfectly conducting plates ($\sigma = \infty$, $\mathbf{E} = 0$) of dimensions w along the y-direction and l along the z-direction lie in the $x = 0$ and $x = d$ planes. The region between the plates is a perfect dielectric ($\sigma = 0$) of material parameters ε and μ. The thickness of the plates is shown exaggerated for convenience in illustration. A potential difference of V_0 is maintained between the plates by connecting a direct voltage source at the end $z = -l$. If fringing of the field due to the finite dimensions of the structure normal to the x-direction is neglected, or, if it is assumed that the structure is part of one which is infinite in extent normal to the x-direction, then the problem can be treated as one-dimensional with x as the variable, and (72) reduces to

$$\frac{d^2\Phi}{dx^2} = 0 \tag{73}$$

The solution for the potential in the charge-free region between the plates is given by

$$\Phi(x) = \frac{V_0}{d}(d - x) \tag{74}$$

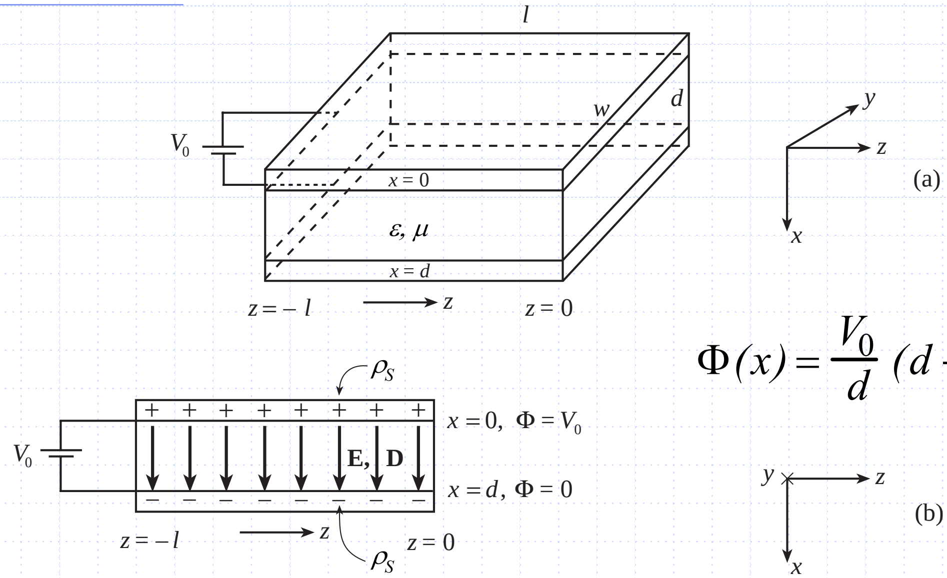
which satisfies (73), as well as the boundary conditions of $\Phi = 0$ at $x = d$ and $\Phi = V_0$ at $x = 0$. The electric field intensity between the plates is then given by

$$\mathbf{E} = -\nabla\Phi = \frac{V_0}{d}\mathbf{a}_x \tag{75}$$

as depicted in the cross-sectional view in Fig. (b) on Slide 50, and resulting in displacement flux density

$$\mathbf{D} = \frac{\varepsilon \mathrm{V}_0}{d}\mathbf{a}_x \tag{76}$$

Example of Parallel-Plate Arrangement; Capacitance



$$\Phi(x) = \frac{V_0}{d}(d - x)$$

Then, using the boundary condition for the normal component of **D** given by (44c) and noting that there is no field inside the conductor, we obtain the magnitude of the charge on either plate to be

$$Q = \left(\frac{\varepsilon V_0}{d}\right)(wl) = \frac{\varepsilon wl}{d} V_0 \tag{77}$$

We can now find the familiar circuit parameter, the capacitance, C, of the parallel-plate arrangement, which is defined as the ratio of the magnitude of the charge on either plate to the potential difference V_0. Thus

$$C = \frac{Q}{V_0} = \frac{\varepsilon wl}{d} \tag{78}$$

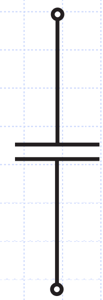
Note that the units of C are the units of ε times meter, that is, farads. The phenomenon associated with the arrangement is that energy is stored in the capacitor in the form of electric field energy between the plates, as given by

$$\begin{aligned} W_e &= \left(\frac{1}{2}\varepsilon E_x^2\right)(wld) \\ &= \frac{1}{2}\left(\frac{\varepsilon wl}{d}\right)V_0^2 \\ &= \frac{1}{2}CV_0^2 \end{aligned} \tag{79}$$

the familiar expression for energy stored in a capacitor.

Electrostatic Analysis of Parallel-Plate Arrangement

$$\mathbf{E} = -\nabla\Phi = \frac{V_0}{d}\,\mathbf{a}_x \qquad \mathbf{D} = \frac{\varepsilon V_0}{d}\,\mathbf{a}_x$$



$$C = \frac{\varepsilon w\,l}{d}$$

$$\mathrm{Q} = \left(\frac{\varepsilon V_0}{d}\right)(wl) = \frac{\varepsilon wl}{d}\,V_0$$

$$C = \frac{Q}{V_0} = \frac{\varepsilon wl}{d}$$

Capacitance of the arrangement, F

$$W_e = \left(\frac{1}{2}\,\varepsilon E_x^2\right)(wld) = \frac{1}{2}\left(\frac{\varepsilon wl}{d}\right)V_0^2 = \frac{1}{2}\,CV_0^2$$

Slide No. 52

Magnetostatic Fields

The equations of interest are (64b) and (64d), or (65b) and (65d). The second of each pair of these equations simply tells us that the magnetostatic field is solenoidal, which as we know holds for any magnetic field, and the first of each pair of these equations enables us, in principle, to determine the magnetostatic field for a given current distribution. Alternatively, the potential function equation (54), which reduces to

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J} \tag{80}$$

can be used to find the magnetic vector potential, **A**, from which the magnetostatic field can be determined by using (47). Equation (80) is the Poisson's equation for the magnetic vector potential, which automatically includes the condition that the field be solenoidal.

Magnetostatic Fields

$$\oint_C \mathbf{H} \cdot d\boldsymbol{l} = \int_S \mathbf{J} \cdot d\mathbf{S}$$

$$\nabla \times \mathbf{H} = \mathbf{J}$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J}$$

Poisson's equation for magnetic vector potential

Slide No. 53

The solution to (80) for a given current density distribution $\mathbf{J}(\mathbf{r})$ is, purely from analogy with the solution (69) to (66), given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\int_{V'} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, dv' \tag{81}$$

Although cast in terms of volume current density, (81) can be formulated in terms of a surface current density, a line current, or a collection of infinitesimal current elements. In particular, for an infinitesimal current element $I\, d\mathbf{l}(\mathbf{r}')$, the solution is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu I\, d\mathbf{l}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{82}$$

It follows from (47) that the magnetic flux density due to the infinitesimal current element is given by

$$\mathbf{B}(\mathbf{r}) = \frac{\mu I\, d\mathbf{l}(\mathbf{r}') \times (\mathbf{r}-\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|^3} \tag{83}$$

which is exactly the law of Biot-Savart that results from Ampere’s force law for the magnetic force between two current elements. Similar to that in the case of (69), (81) or its alternate forms can be used to find the magnetic vector potential and then the magnetic field by using (47) for a specified current distribution.

In a current-free region, $\mathbf{J} = 0$, and (80) reduces to

$$\nabla^2\mathbf{A} = 0 \tag{84}$$

The field is then due to currents outside the region, such as surface currents on conductors bounding the region. The situation is then one of solving a boundary value problem as in the case of (72). However, since the boundary condition (44b) relates the

Solution for Vector Potential and Field

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \int_{V'} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, dv'$$

Solution for current distribution

$$\mathbf{A}(\mathbf{r}) = \frac{\mu I \, d\mathbf{l}(\mathbf{r}')}{4\pi |\mathbf{r} - \mathbf{r}'|}$$

Solution for current element

$$\mathbf{B}(\mathbf{r}) = \frac{\mu I \, d\mathbf{l}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{4\pi |\mathbf{r} - \mathbf{r}'|^3}$$

Magnetic field due to current element

$$\nabla^2 \mathbf{A} = 0$$

For current-free region

magnetic field directly to the surface current density, it is straightforward and more convenient to determine the magnetic field directly by using (65b) and (65d).

Slide Nos. 54-56

A simple example is that of the parallel-plate arrangement of Fig. (a) on Slide 50 with the plates connected by another conductor at the end $z = 0$ and driven by a source of direct current I_0 at the end $z = -l$, as shown in Fig. (a) on Slide 54. If fringing of the field due to the finite dimensions of the structure normal to the x-direction is neglected, or, if it is assumed that the structure is part of one which is infinite in extent normal to the x-direction, then the problem can be treated as one-dimensional with x as the variable and we can write the current density on the plates to be

$$\mathbf{J}_S = \begin{cases} (I_0/w)\mathbf{a}_z & \textit{on the plate } x = 0 \\ (I_0/w)\mathbf{a}_x & \textit{on the plate } z = 0 \\ -(I_0/w)\mathbf{a}_z & \textit{on the plate } x = d \end{cases} \tag{85}$$

In the current-free region between the plates, (65b) reduces to

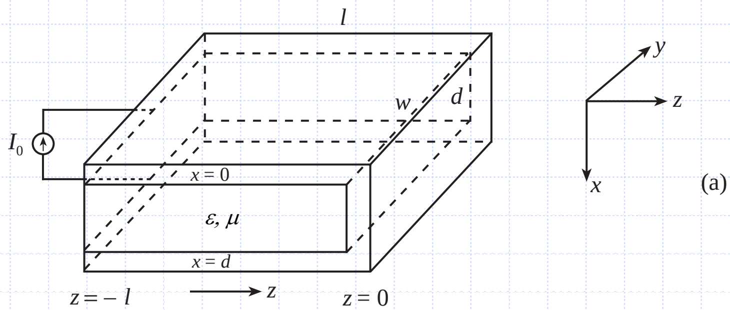
$$\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & 0 & 0 \\ H_x & H_y & H_z \end{vmatrix} = 0 \tag{86}$$

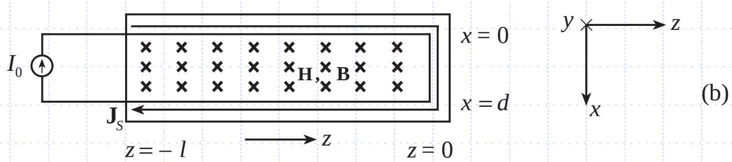
and (65d) reduces to

$$\frac{\partial B_x}{\partial x} = 0 \tag{87}$$

so that each component of the field, if it exists, has to be uniform. This automatically forces H_x and H_z to be zero since nonzero value of these components do not satisfy the boundary conditions (44b) and (44d) on the plates, keeping in mind that the field is

Example of Parallel-Plate Arrangement; Inductance





entirely in the region between the conductors. Thus, as depicted in the cross-sectional view in the figure,

$$\mathbf{H} = \frac{I_0}{w}\,\mathbf{a}_y \tag{88}$$

which satisfies the boundary condition (44b) on all three plates, and results in magnetic flux density

$$\mathbf{B} = \frac{\mu I_0}{w}\,\mathbf{a}_y \tag{89}$$

The magnetic flux, ψ, linking the current I_0, is then given by

$$\psi = \left(\frac{\mu I_0}{w}\right)(dl) = \left(\frac{\mu dl}{w}\right) I_0 \tag{90}$$

We can now find the familiar circuit parameter, the inductance, L, of the parallel-plate arrangement, which is defined as the ratio of the magnetic flux linking the current to the current. Thus

$$L = \frac{\psi}{I_0} = \frac{\mu dl}{w} \tag{91}$$

Note that the units of L are the units of μ times meter, that is, henrys. The phenomenon associated with the arrangement is that energy is stored in the inductor in the form of magnetic field energy between the plates, as given by

$$\begin{aligned} W_m &= \left(\frac{1}{2}\mu H^2\right)(wld) \\ &= \frac{1}{2}\left(\frac{\mu dl}{w}\right) I_0^2 \\ &= \frac{1}{2} L I_0^2 \end{aligned} \tag{92}$$

the familiar expression for energy stored in an inductor.

Magnetostatic Analysis of Parallel-Plate Arrangement

$$\mathbf{J}_S = \begin{cases} (I_0/w)\mathbf{a}_z & \text{on the plate } x = 0 \\ (I_0/w)\mathbf{a}_x & \text{on the plate } z = 0 \\ -(I_0/w)\mathbf{a}_z & \text{on the plate } x = d \end{cases}$$

$$\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & 0 & 0 \\ H_x & H_y & H_z \end{vmatrix} = 0$$

$$\frac{\partial B_x}{\partial x} = 0$$

$$\mathbf{H} = \frac{I_0}{w}\mathbf{a}_y$$

Magnetostatic Analysis of Parallel-Plate Arrangement

$$\mathbf{B} = \frac{\mu I_0}{w}\mathbf{a}_y$$

$$L = \frac{\mu\, dl}{w}$$

$$\psi = \left(\frac{\mu I_0}{w}\right)(dl) = \left(\frac{\mu dl}{w}\right) I_0$$

$$L = \frac{\psi}{I_0} = \frac{\mu dl}{w}$$

Inductance of the arrangement, H

$$W_m = \left(\frac{1}{2}\mu H^2\right)(wld) = \frac{1}{2}\left(\frac{\mu dl}{w}\right) I_0^2 = \frac{1}{2} L I_0^2$$

Slide No. 57

Electromagnetostatic Fields

The equations of interest are

$$\oint_C \mathbf{E} \bullet d\mathbf{l} = 0 \quad (93a)$$

$$\oint_C \mathbf{H} \bullet d\mathbf{l} = \int_S \mathbf{J}_c \bullet d\mathbf{S} = \sigma \int_S \mathbf{E} \bullet d\mathbf{S} \quad (93b)$$

$$\oint_S \mathbf{D} \bullet d\mathbf{S} = 0 \quad (93c)$$

$$\oint_S \mathbf{B} \bullet d\mathbf{S} = 0 \quad (93d)$$

or, in differential form

$$\nabla \times \mathbf{E} = 0 \quad (94a)$$

$$\nabla \times \mathbf{H} = \mathbf{J}_c = \sigma \mathbf{E} \quad (94b)$$

$$\nabla \bullet \mathbf{D} = 0 \quad (94c)$$

$$\nabla \bullet \mathbf{B} = 0 \quad (94d)$$

From (94a) and (94c), we note that Laplace's equation (72) for the electrostatic potential is satisfied, so that, for a given problem, the electric field can be found in the same manner as in the case of the example of the capacitor arrangement with perfect dielectric. The magnetic field is then found by using (94b), and making sure that (94d) is also satisfied.

Electromagnetostatic Fields

$$(\mathbf{J} = \mathbf{J}_c = \sigma \mathbf{E})$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \qquad \nabla \times \mathbf{E} = 0$$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J}_c \cdot d\mathbf{S} = \sigma \int_S \mathbf{E} \cdot d\mathbf{S} \qquad \nabla \times \mathbf{H} = \mathbf{J}_c = \sigma \mathbf{E}$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = 0 \qquad \nabla \cdot \mathbf{D} = 0$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

Slide Nos. 58-60

A simple example is that of the parallel-plate arrangement of Fig. (a) on Slide 50, but with an imperfect dielectric material of parameters σ, ε, and μ, between the plates, as shown in Fig (a) on Slide 58. Then, the electric field between the plates is the same as that given by (75), that is,

$$\mathbf{E} = \frac{V_0}{d}\mathbf{a}_x \tag{95}$$

resulting in a conduction current of density

$$\mathbf{J}_c = \frac{\sigma V_0}{d}\mathbf{a}_x \tag{96}$$

from the top plate to the bottom plate, as depicted in the cross-sectional view of Fig. (b) on Slide 58. Since $\partial\rho/\partial t = 0$ at the boundaries between the plates and the slab, continuity of current is satisfied by the flow of surface current on the plates. At the input $z = -l$, this surface current, which is the current drawn from the source, must be equal to the total current flowing from the top to the bottom plate. It is given by

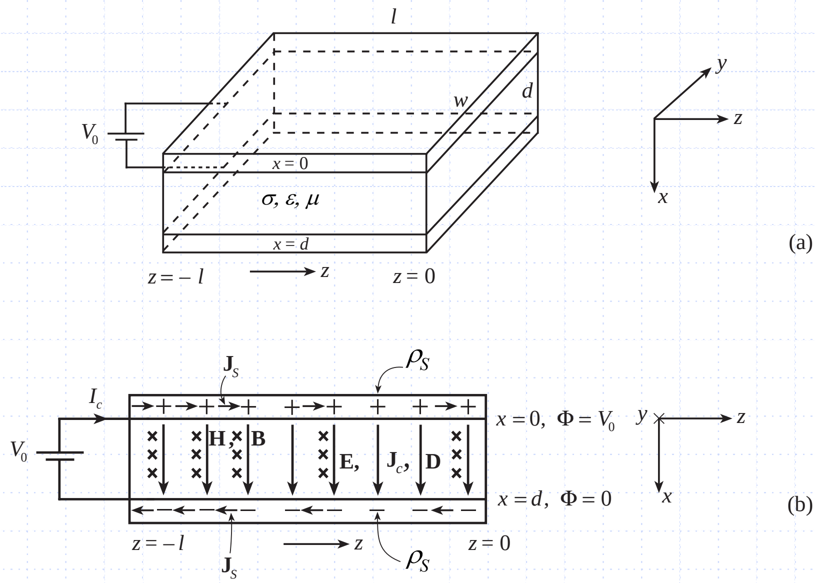
$$I_c = \left(\frac{\sigma V_0}{d}\right)(wl) = \frac{\sigma wl}{d}V_0 \tag{97}$$

We can now find the familiar circuit parameter, the conductance, G, of the parallel-plate arrangement, which is defined as the ratio of the current drawn from the source to the source voltage V_0. Thus

$$G = \frac{I_c}{V_0} = \frac{\sigma wl}{d} \tag{98}$$

Note that the units of G are the units of σ times meter, that is, siemens(S). The reciprocal quantity, R, the resistance of the parallel-plate arrangement, is given by

Example of Parallel-Plate Arrangement



$$R = \frac{V_0}{I_c} = \frac{d}{\sigma w l} \tag{99}$$

The unit of R is ohms. The phenomenon associated with the arrangement is that power is dissipated in the material between the plates, as given by

$$\begin{aligned} P_d &= (\sigma E^2)(wld) \\ &= \left(\frac{\sigma w l}{d}\right) V_0^2 \\ &= G V_0^2 \\ &= \frac{V_0^2}{R} \end{aligned} \tag{100}$$

the familiar expression for power dissipated in a resistor.

Electromagnetostatic Analysis of Parallel-Plate Arrangement

$$\mathbf{E} = \frac{V_0}{d}\mathbf{a}_x$$

$$\mathbf{J}_c = \frac{\sigma V_0}{d}\mathbf{a}_x$$

$$I_c = \left(\frac{\sigma V_0}{d}\right)(wl) = \frac{\sigma wl}{d} V_0$$

Electromagnetostatic Analysis of Parallel-Plate Arrangement

$$G = \frac{I_c}{V_0} = \frac{\sigma w l}{d}$$ Conductance, S

$$R = \frac{V_0}{I_c} = \frac{d}{\sigma w l}$$ Resistance, ohms

$$P_d = (\sigma E^2)(wld) = \left(\frac{\sigma w l}{d}\right) V_0^2$$

$$= G V_0^2 = \frac{V_0^2}{R}$$

Slide No. 61

Proceeding further, we find the magnetic field between the plates by using (94b), and noting that the geometry of the situation requires a y-component of **H**, dependent on z, to satisfy the equation. Thus

$$\mathbf{H} = H_y(z)\mathbf{a}_y \tag{101a}$$

$$\frac{\partial H_y}{\partial z} = -\frac{\sigma V_0}{d} \tag{101b}$$

$$\mathbf{H} = -\frac{\sigma V_0}{d} z\mathbf{a}_y \tag{101c}$$

where the constant of integration is set to zero, since the boundary condition at $z = 0$ requires H_y to be zero for z equal to zero. Note that the magnetic field is directed in the positive y-direction (since z is negative) and increases linearly from $z = 0$ to $z = -l$, as depicted in Fig. (b) on Slide 58. It also satisfies the boundary condition at $z = -l$ by being consistent with the current drawn from the source to be $w\left[H_y\right]_{z=-l} = (\sigma V_0/d)(wl) = I_c$.

Because of the existence of the magnetic field, the arrangement is characterized by an inductance, which can be found either by using the flux linkage concept, or by the energy method. To use the flux linkage concept, we recognize that a differential amount of magnetic flux $d\psi' = \mu H_y d(dz')$ between z equal to $(z' - dz')$ and z equal to z', where $-l < z' < 0$, links only that part of the current that flows from the top plate to the bottom plate between $z = z'$ and $z = 0$, thereby giving a value of $(-z'/l)$ for the fraction, N, of the total current linked. Thus, the inductance, familiarly known as the internal inductance, denoted L_i, since it is due to magnetic field internal to the current distribution, as

Electromagnetostatic Analysis of Parallel-Plate Arrangement

$$\mathbf{H} = H_y(z)\,\mathbf{a}_y \qquad \frac{\partial H_y}{\partial z} = -\frac{\sigma V_0}{d}$$

$$\mathbf{H} = -\frac{\sigma V_0}{d} z\,\mathbf{a}_y$$

$$L_i = \frac{1}{I_c}\int_{z'=-l}^{0} N\,d\psi' = \frac{1}{I_c}\int_{z'=-l}^{0}\left(\frac{-z'}{l}\right)[\mu H_y d(dz')]$$

$$= \frac{1}{3}\frac{\mu d l}{w}$$

Internal Inductance

compared to that in (91) for which the magnetic field is external to the current distribution, is given by

$$\begin{aligned} L_i &= \frac{1}{I_c} \int_{z'=-l}^{0} N \, d\psi' \\ &= \frac{1}{3} \frac{\mu dl}{w} \end{aligned} \tag{102}$$

or, 1/3 times the inductance of the structure if $\sigma = 0$ and the plates are joined at $z = 0$, as in the case of the arrangement of parallel-plates connected at the end.

Slide No. 62

Alternatively, if the energy method is used by computing the energy stored in the magnetic field and setting it equal to $\frac{1}{2} L_i I_c^2$, then we have

$$\begin{aligned} L_i &= \frac{1}{I_c^2} (dw) \int_{z=-l}^{0} \mu H_y^2 \, dz \\ &= \frac{1}{3} \frac{\mu dl}{w} \end{aligned} \tag{103}$$

same as in (102).

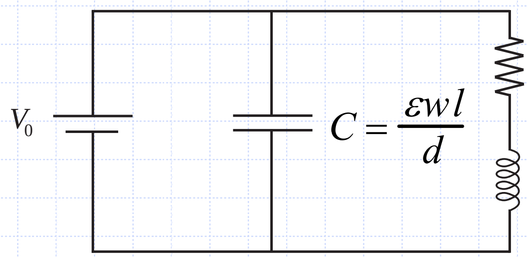
Finally, recognizing that there is energy storage associated with the electric field between the plates, we note that the arrangement has also associated with it a capacitance C, equal to $\varepsilon wl/d$. Thus, all three properties of conductance, capacitance, and inductance are associated with the structure. Since for $\sigma = 0$ the situation reduces to that of the single capacitor arrangement, we can represent the arrangement to be equivalent to the circuit shown. Note that the capacitor is charged to the voltage V_0 and the current through it is zero (open circuit condition). The voltage across the inductor is zero (short circuit condition) and the current through it is V_0/R. Thus, the current drawn from the voltage source is V_0/R and the voltage source views a single resistor R, as far as the current drawn from it is concerned.

Electromagnetostatic Analysis of Parallel-Plate Arrangement

Alternatively,

$$L_i = \frac{1}{I_c^2}\,(dw)\int_{z=-l}^{0} \mu H_y^2 \, dz = \frac{1}{3}\frac{\mu d l}{w}$$

Equivalent Circuit



$$R = \frac{d}{\sigma w l}$$

$$L_i = \frac{1}{3}\frac{\mu d l}{w}$$

Slide No. 63

<u>Quasistatic Fields</u>

As already mentioned, quasistatic fields are a class of dynamic fields for which certain features can be analyzed as though the fields were static. In terms of behavior in the frequency domain, they are low-frequency extensions of static fields present in a physical structure, when the frequency of the source driving the structure is zero, or low-frequency approximations of time-varying fields in the structure that are complete solutions to Maxwell's equations. Here, we use the approach of low-frequency extensions of static fields. Thus, for a given structure, we begin with a time-varying field having the same spatial characteristics as that of the static field solution for the structure, and obtain field solutions containing terms up to and including the first power (which is the lowest power) in ω for their amplitudes. Depending on whether the predominant static field is electric or magnetic, quasistatic fields are called electroquasistatic fields or magnetoquasistatic fields. We shall now consider these separately.

Quasistatic Fields

For quasistatic fields, certain features can be analyzed as though the fields were static. In terms of behavior in the frequency domain, they are low-frequency extensions of static fields present in a physical structure, when the frequency of the source driving the structure is zero, or low-frequency approximations of time-varying fields in the structure that are complete solutions to Maxwell's equations. Here, we use the approach of low-frequency extensions of static fields. Thus, for a given structure, we begin with a time-varying field having the same spatial characteristics as that of the static field solution for the structure and obtain field solutions containing terms up to and including the first power (which is the lowest power) in ω for their amplitudes.

Slide Nos. 64-65

<u>Electroquasistatic Fields</u>

For electroquasistatic fields, we begin with the electric field having the spatial dependence of the static field solution for the given arrangement. An example is provided by the parallel-plate arrangement with perfect dielectric (Slide 50), excited by a sinusoidally time-varying voltage source $V_g(t) = V_0 \cos \omega t$, instead of a direct voltage source. Then,

$$\mathbf{E}_0 = \frac{V_0}{d} \cos \omega t \; \mathbf{a}_x \tag{104}$$

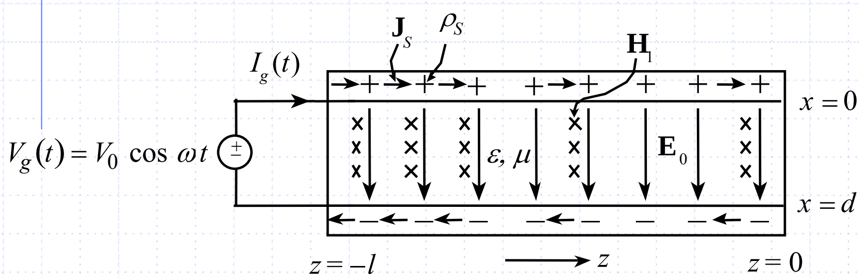
where the subscript 0 denotes that the amplitude of the field is of the zeroth power in ω. This results in a magnetic field in accordance with Maxwell's equation for the curl of **H**, given by (28). Thus, noting that $\mathbf{J} = 0$ in view of the perfect dielectric medium, we have for the geometry of the arrangement,

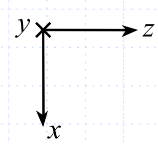
$$\frac{\partial H_{y1}}{\partial z} = -\frac{\partial D_{x0}}{\partial t} = \frac{\omega \varepsilon V_0}{d} \sin \omega t \tag{105}$$

$$\mathbf{H}_1 = \frac{\omega \varepsilon V_0 z}{d} \sin \omega t \; \mathbf{a}_y \tag{106}$$

where we have also satisfied the boundary condition at $z = 0$ by choosing the constant of integration such that $\left[H_{y1}\right]_{z=0}$ is zero, and the subscript 1 denotes that the amplitude of the field is of the first power in ω. Note that the amplitude of H_{y1} varies linearly with z, from zero at $z = 0$ to a maximum at $z = -l$.

Electroquasistatic Fields





Electroquasistatic Analysis of Parallel-Plate Arrangement

$$\mathbf{E}_0 = \frac{V_0}{d} \cos \omega t \ \mathbf{a}_x$$

$$\frac{\partial H_{y1}}{\partial z} = -\frac{\partial D_{x0}}{\partial t} = \frac{\omega \varepsilon V_0}{d} \sin \omega t$$

$$\mathbf{H}_1 = \frac{\omega \varepsilon V_0 z}{d} \sin \omega t \ \mathbf{a}_y$$

Slide No. 66

We stop the solution here, because continuing the process by substituting (106) into Maxwell's curl equation for **E**, (27), to obtain the resulting electric field will yield a term having amplitude proportional to the second power in ω. This simply means that the fields given as a pair by (104) and (106) do not satisfy (27), and hence are not complete solutions to Maxwell's equations. The complete solutions are obtained by solving Maxwell's equations simultaneously and subject to the boundary conditions for the given problem.

Proceeding further, we obtain the current drawn from the voltage source to be

$$\begin{aligned} I_g(t) &= w\left[H_{y1}\right]_{z=-l} \\ &= -\omega\left(\frac{\varepsilon wl}{d}\right)V_0 \sin \omega t \\ &= C\frac{dV_g(t)}{dt} \end{aligned} \qquad (107)$$

or,

$$\bar{I}_g = j\omega C\bar{V}_g \qquad (108)$$

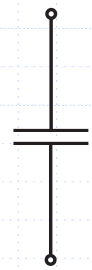
where $C = (\varepsilon wl/d)$ is the capacitance of the arrangement obtained from static field considerations. Thus, the input admittance of the structure is $j\omega C$ so that its low frequency input behavior is essentially that of a single capacitor of value same as that found from static field analysis of the structure.

Electroquasistatic Analysis of Parallel-Plate Arrangement

$$I_g(t) = w\left[H_{y1}\right]_{z=-l}$$

$$= -\omega\left(\frac{\varepsilon w l}{d}\right) V_0 \sin \omega t$$

$$= C\frac{dV_g(t)}{dt}$$



$$C = \frac{\varepsilon w l}{d}$$

$$\bar{I}_g = j\omega C \bar{V}_g \quad \text{where} \quad C = \frac{\varepsilon w l}{d}$$

Slide No. 67

Indeed, from considerations of power flow, using Poynting's theorem, we obtain the power flowing into the structure to be

$$\begin{aligned} P_{\text{in}} &= wd\big[E_{x0}H_{y1}\big]_{z=0} \\ &= -\left(\frac{\varepsilon wl}{d}\right)\omega V_0^2 \sin\omega t \cos\omega t \\ &= \frac{d}{dt}\left(\frac{1}{2}CV_g^2\right) \end{aligned} \tag{109}$$

which is consistent with the electric energy stored in the structure for the static case, as given by (79).

Electroquasistatic Analysis of Parallel-Plate Arrangement

$$P_{\text{in}} = wd\left[E_{x0}H_{y1}\right]_{z=0}$$

$$= -\left(\frac{\varepsilon wl}{d}\right)\omega V_0^2 \sin\omega t \cos\omega t$$

$$= \frac{d}{dt}\left(\frac{1}{2}CV_g^2\right)$$

Slide Nos. 68-69

<u>Magnetoquasistatic Fields</u>

For magnetoquasistatic fields, we begin with the magnetic field having the spatial dependence of the static field solution for the given arrangement. An example is provided by the parallel-plate arrangement with perfect dielectric and connected at the end (Slide 54), excited by a sinusoidally time-varying current source $I_g(t) = I_0 \cos \omega t$, instead of a direct current source. Then,

$$\mathbf{H}_0 = \frac{I_0}{w} \cos \omega t \, \mathbf{a}_y \tag{110}$$

where the subscript 0 again denotes that the amplitude of the field is of the zeroth power in ω. This results in an electric field in accordance with Maxwell's curl equation for **E**, given by (27). Thus, we have for the geometry of the arrangement,

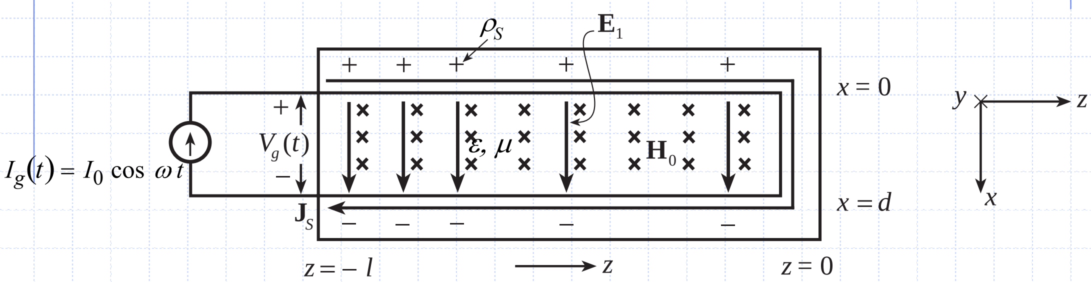
$$\frac{\partial E_{x1}}{\partial z} = -\frac{\partial B_{y0}}{\partial t} = \frac{\omega \mu I_0}{w} \sin \omega t \tag{111}$$

$$\mathbf{E}_1 = \frac{\omega \mu I_0 z}{w} \sin \omega t \, \mathbf{a}_x \tag{112}$$

where we have also satisfied the boundary condition at $z = 0$ by choosing the constant of integration such that $[E_{x1}]_{z=0} = 0$ is zero, and again the subscript 1 denotes that the amplitude of the field is of the first power in ω. Note that the amplitude of E_{x1} varies linearly with z, from zero at $z = 0$ to a maximum at $z = -l$.

As is the case of electroquasistatic fields, we stop the process here, because continuing it by substituting (112) into Maxwell's curl equation for **H**, (28), to obtain the resulting magnetic field will yield a term having amplitude proportional to the second power in ω. This simply means that the fields given as a pair by (110) and (112) do not

Magnetoquasistatic Fields



satisfy (28), and hence are not complete solutions to Maxwell's equations. The complete solutions are obtained by solving Maxwell's equations simultaneously and subject to the boundary conditions for the given problem.

Magnetoquasistatic Analysis of Parallel-Plate Arrangement

$$\mathbf{H}_0 = \frac{I_0}{w} \cos \omega t \, \mathbf{a}_y$$

$$\frac{\partial E_{x1}}{\partial z} = -\frac{\partial B_{y0}}{\partial t} = \frac{\omega \mu I_0}{w} \sin \omega t$$

$$\mathbf{E}_1 = \frac{\omega \mu I_0 z}{w} \sin \omega t \, \mathbf{a}_x$$

Slide No. 70

Proceeding further, we obtain the voltage across the current source to be

$$\begin{aligned} V_g(t) &= d\left[E_{x1}\right]_{z=-l} \\ &= -\omega\left(\frac{\mu dl}{w}\right) I_0 \ \sin\ \omega t \\ &= L\frac{dI_g(t)}{dt} \end{aligned} \tag{113}$$

or

$$\bar{V}_g = j\omega L \bar{I}_g \tag{114}$$

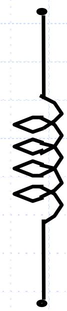
where $L = (\mu dl/w)$ is the inductance of the arrangement obtained from static field considerations. Thus, the input impedance of the structure is $j\omega L$, such that its low frequency input behavior is essentially that of a single inductor of value same as that found from static field analysis of the structure.

Magnetoquasistatic Analysis of Parallel-Plate Arrangement

$$V_g(t) = d\left[E_{x1}\right]_{z=-l}$$

$$= -\omega\left(\frac{\mu dl}{w}\right) I_0 \sin \omega t$$

$$= L\,\frac{dI_g(t)}{dt}$$



$$L = \frac{\mu dl}{w}$$

$\bar{V}_g = j\omega L\, \bar{I}_g$ where $L = \dfrac{\mu dl}{w}$

Slide No. 71

Indeed, from considerations of power flow, using Poynting's theorem, we obtain the power flowing into the structure to be

$$\begin{aligned} P_{\text{in}} &= wd\left[E_{x1}H_{y0}\right]_{z=-l} \\ &= -\left(\frac{\mu dl}{w}\right)\omega\, I_0^2 \,\sin\,\omega t\,\cos\,\omega t \\ &= \frac{d}{dt}\left(\frac{1}{2}LI_g^2\right) \end{aligned} \tag{115}$$

which is consistent with the magnetic energy stored in the structure for the static case, as given by (92).

Magnetoquasistatic Analysis of Parallel-Plate Arrangement

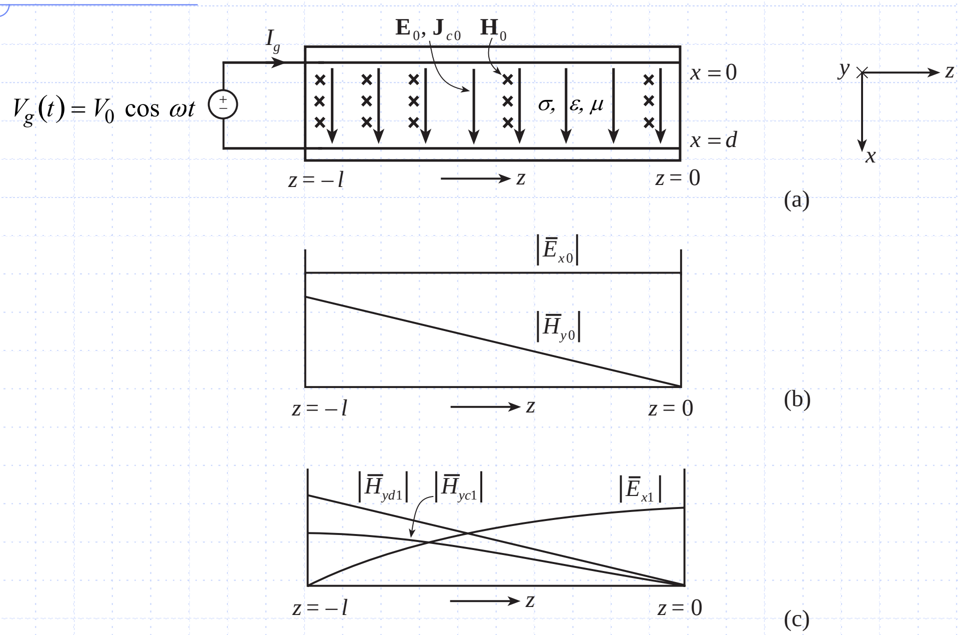
$$
\begin{aligned}
P_{\text{in}} &= wd\left[E_{x1}H_{y0}\right]_{z=-l} \\
&= -\left(\frac{\mu d l}{w}\right)\omega I_0^2 \sin\omega t \cos\omega t \\
&= \frac{d}{dt}\left(\frac{1}{2}LI_g^2\right)
\end{aligned}
$$

Slide No. 72

Quasistatic Fields in a Conductor

If the dielectric slab in the arrangement of Slide 64 is conductive, then both electric and magnetic fields exist in the static case, because of the conduction current, as discussed under electromagnetostatic fields. Furthermore, the electric field of amplitude proportional to the first power in ω contributes to the creation of magnetic field of amplitude proportional to the first power in ω, in addition to that from electric field of amplitude proportional to the zeroth power in ω.

Quasistatic Fields in a Conductor



Slide No. 73

Thus, using the results from the static field analysis from the arrangement of Slide 58, we have for this arrangement,

$$\mathbf{E}_0 = \frac{V_0}{d}\cos\omega t\ \mathbf{a}_x \tag{116}$$

$$\mathbf{J}_{c0} = \sigma\mathbf{E}_0 = \frac{\sigma V_0}{d}\cos\omega t\ \mathbf{a}_x \tag{117}$$

$$\mathbf{H}_0 = -\frac{\sigma V_0 z}{d}\cos\omega t\ \mathbf{a}_y \tag{118}$$

as depicted in the figure. Also, the variations with z of the amplitudes of E_{x0} and H_{y0} are shown in Fig. (b) on Slide 72.

The magnetic field given by (118) gives rise to an electric field having amplitude proportional to the first power in ω, in accordance with Maxwell's curl equation for **E**, (27). Thus

$$\frac{\partial E_{x1}}{\partial z} = -\frac{\partial B_{y0}}{\partial t} = -\frac{\omega\mu\sigma V_0 z}{d}\sin\,\omega t \tag{119}$$

$$E_{x1} = -\frac{\omega\mu\sigma V_0}{2d}\left(z^2 - l^2\right)\sin\,\omega t \tag{120}$$

where we have also made sure that the boundary condition at $z = -l$ is satisfied. This boundary condition requires that E_x be equal to V_g/d at $z = -l$. Since this is satisfied by E_{x0} alone, it follows that E_{x1} must be zero at $z = -l$.

Quasistatic Analysis of Parallel-Plate Arrangement with Conductor

$$\mathbf{E}_0 = \frac{V_0}{d}\cos\omega t\ \mathbf{a}_x$$

$$\mathbf{J}_{c0} = \sigma\mathbf{E}_0 = \frac{\sigma V_0}{d}\cos\omega t\ \mathbf{a}_x$$

$$\mathbf{H}_0 = -\frac{\sigma V_0 z}{d}\cos\omega t\ \mathbf{a}_y$$

$$\frac{\partial E_{x1}}{\partial z} = -\frac{\partial B_{y0}}{\partial t} = -\frac{\omega\mu\sigma V_0 z}{d}\sin\omega t$$

$$E_{x1} = -\frac{\omega\mu\sigma V_0}{2d}\left(z^2 - l^2\right)\sin\omega t$$

Slide No. 74

The electric field given by (116) and that given by (120) together give rise to a magnetic field having terms with amplitudes proportional to the first power in ω, in accordance with Maxwell's curl equation for **H**, (28). Thus

$$\begin{aligned}\frac{\partial H_{y1}}{\partial z} &= -\sigma E_{x1} - \varepsilon \frac{\partial E_{x0}}{\partial t} \\ &= \frac{\omega\mu\sigma^2 V_0}{2d}\left(z^2 - l^2\right)\sin\,\omega t + \frac{\omega\varepsilon V_0}{d}\sin\,\omega t\end{aligned} \tag{121}$$

$$H_{y1} = \frac{\omega\mu\sigma^2 V_0\left(z^3 - 3zl^2\right)}{6d}\sin\,\omega t + \frac{\omega\varepsilon V_0 z}{d}\sin\,\omega t \tag{122}$$

where we have also made sure that the boundary condition at $z = 0$ is satisfied. This boundary condition requires that H_y be equal to zero at $z = 0$, which means that all of its terms must be zero at $z = 0$. Note that the first term on the right side of (122) is the contribution from the conduction current in the material resulting from E_{x1} and the second term is the contribution from the displacement current resulting from E_{x0}. Denoting these to be H_{yc1} and H_{yd1}, respectively, we show the variations with z of the amplitudes of all the field components having amplitudes proportional to the first power in ω, in Fig. (c) on Slide 72.

Quasistatic Analysis of Parallel-Plate Arrangement with Conductor

$$\frac{\partial H_{y1}}{\partial z} = -\sigma E_{x1} - \varepsilon \frac{\partial E_{x0}}{\partial t}$$

$$= \frac{\omega \mu \sigma^2 V_0}{2d}\left(z^2 - l^2\right)\sin \omega t + \frac{\omega \varepsilon V_0}{d} \sin \omega t$$

$$H_{y1} = \frac{\omega \mu \sigma^2 V_0\left(z^3 - 3zl^2\right)}{6d}\sin \omega t + \frac{\omega \varepsilon V_0 z}{d} \sin \omega t$$

Slide No. 75

Now, adding up the contributions to each field, we obtain the total electric and magnetic fields up to and including the terms with amplitudes proportional to the first power in ω to be

$$E_x = \frac{V_0}{d}\cos\omega t - \frac{\omega\mu\sigma V_0}{2d}\left(z^2 - l^2\right)\sin\omega t \tag{123a}$$

$$H_y = -\frac{\sigma V_0 z}{d}\cos\omega t + \frac{\omega\varepsilon V_0 z}{d}\sin\omega t + \frac{\omega\mu\sigma^2 V_0\left(z^3 - 3zl^2\right)}{6d}\sin\omega t \tag{123b}$$

or

$$\overline{E}_x = \frac{\overline{V}_g}{d} + j\omega\frac{\mu\sigma}{2d}\left(z^2 - l^2\right)\overline{V}_g \tag{124a}$$

$$\overline{H}_y = -\frac{\sigma z}{d}\overline{V}_g - j\omega\frac{\varepsilon z}{d}\overline{V}_g - j\omega\frac{\mu\sigma^2\left(z^3 - 3zl^2\right)}{6d}\overline{V}_g \tag{124b}$$

Quasistatic Analysis of Parallel-Plate Arrangement with Conductor

$$E_x = \frac{V_0}{d}\cos\omega t - \frac{\omega\mu\sigma V_0}{2d}\left(z^2 - l^2\right)\sin\omega t$$

$$H_y = -\frac{\sigma V_0 z}{d}\cos\omega t + \frac{\omega\varepsilon V_0 z}{d}\sin\omega t + \frac{\omega\mu\sigma^2 V_0\left(z^3 - 3zl^2\right)}{6d}\sin\omega t$$

$$\bar{E}_x = \frac{\bar{V}_g}{d} + j\omega\frac{\mu\sigma}{2d}\left(z^2 - l^2\right)\bar{V}_g$$

$$\bar{H}_y = -\frac{\sigma z}{d}\bar{V}_g - j\omega\frac{\varepsilon z}{d}\bar{V}_g - j\omega\frac{\mu\sigma^2\left(z^3 - 3zl^2\right)}{6d}\bar{V}_g$$

Slide No. 76

Finally, the current drawn from the voltage source is given by

$$\begin{aligned} \bar{I}_g &= w\left[\bar{H}_y\right]_{z=-l} \\ &= \left(\frac{\sigma wl}{d} + j\omega\frac{\varepsilon wl}{d} - j\omega\frac{\mu\sigma^2 wl^3}{3d}\right)\bar{V}_g \end{aligned} \tag{125}$$

The input admittance of the structure is given by

$$\begin{aligned} \bar{Y}_{\text{in}} &= \frac{\bar{I}_g}{\bar{V}_g} = j\omega\frac{\varepsilon wl}{d} + \frac{\sigma wl}{d}\left(1 - j\omega\frac{\mu\sigma l^2}{3}\right) \\ &\approx j\omega\frac{\varepsilon wl}{d} + \frac{1}{\frac{d}{\sigma wl}\left(1 + j\omega\frac{\mu\sigma l^2}{3}\right)} \end{aligned} \tag{126}$$

where we have used the approximation $[1 + j\omega(\mu\sigma l^2/3)]^{-1} \approx [1 - j\omega(\mu\sigma l^2/3)]$.

Quasistatic Analysis of Parallel-Plate Arrangement with Conductor

$$\bar{I}_g = w\left[\bar{H}_y\right]_{z=-l}$$

$$= \left(\frac{\sigma w l}{d} + j\omega \frac{\varepsilon w l}{d} - j\omega \frac{\mu\sigma^2 w l^3}{3d} \right) \bar{V}_g$$

$$\bar{Y}_{\text{in}} = \frac{\bar{I}_g}{\bar{V}_g} = j\omega \frac{\varepsilon w l}{d} + \frac{\sigma w l}{d}\left(1 - j\omega \frac{\mu\sigma l^2}{3}\right)$$

$$\approx j\omega \frac{\varepsilon w l}{d} + \frac{1}{\frac{d}{\sigma w l}\left(1 + j\omega \frac{\mu\sigma l^2}{3}\right)}$$

Slide No. 77

Proceeding further,

we have

$$\begin{aligned}\bar{Y}_{\text{in}} &= j\omega\frac{\varepsilon wl}{d} + \frac{1}{\frac{d}{\sigma wl} + j\omega\frac{\mu dl}{3w}} \\ &= j\omega C + \frac{1}{R + j\omega L_i}\end{aligned} \tag{127}$$

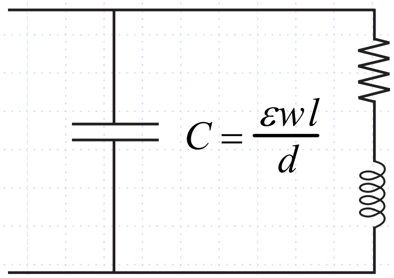
where $C = \varepsilon wl/d$ is the capacitance of the structure if the material is a perfect dielectric, $R = d/\sigma wl$ is the resistance of the structure, and $L_i = \mu dl/3w$ is the internal inductance of the structure, all computed from static field analysis of the structure.

The equivalent circuit corresponding to (127) consists of capacitance C in parallel with the series combination of resistance R and internal inductance L_i, same as that on Slide 62. Thus, the low-frequency input behavior of the structure is essentially the same as that of the equivalent circuit on Slide 62, with the understanding that its input admittance must also be approximated to first-order terms. Note that for $\sigma = 0$, the input admittance of the structure is purely capacitive. For nonzero σ, a critical value of σ equal to $\sqrt{3\varepsilon/\mu l^2}$ exists for which the input admittance is purely conductive. For values of σ smaller than the critical value, the input admittance is complex and capacitive, and for values of σ larger than the critical value, the input admittance is complex and inductive.

Quasistatic Analysis of Parallel-Plate Arrangement with Conductor

$$\bar{Y}_{\text{in}} = j\omega \frac{\varepsilon w l}{d} + \frac{1}{\frac{d}{\sigma w l} + j\omega \frac{\mu d l}{3w}} = j\omega C + \frac{1}{R + j\omega L_i}$$

Equivalent Circuit



$$C = \frac{\varepsilon w l}{d}$$

$$R = \frac{d}{\sigma w l}$$

$$L_i = \frac{1}{3}\frac{\mu d l}{w}$$

Slide No. 78

We have seen that quasistatic field analysis of a physical structure provides information concerning the low-frequency input behavior of the structure. As the frequency is increased beyond that for which the quasistatic approximation is valid, terms in the infinite series solutions for the fields beyond the first-order terms need to be included. While one can obtain equivalent circuits for frequencies beyond the range of validity of the quasistatic approximation by evaluating the higher order terms, no further insight is gained through that process and it is more straightforward to obtain the exact solution by resorting to simultaneous solution of Maxwell's equations, when a closed form solution is possible.

Waves and the Distributed Circuit Concept

We have seen that quasistatic field analysis of a physical structure Provides information concerning the low-frequency input behavior of the structure. As the frequency is increased beyond that for which the quasistatic approximation is valid, terms in the infinite series solutions for the fields beyond the first-order terms need to be included. While one can obtain equivalent circuits for frequencies beyond the range of validity of the quasistatic approximation by evaluating the higher order terms, no further insight is gained through that process, and it is more straight-forward to obtain the exact solution by resorting to simultaneous solution of Maxwell's equations when a closed form solution is possible.

Slide No. 79

<u>Wave Equation</u>

Let us, for simplicity, consider the structures on Slides 64 and 68, for which the material between the plates is a perfect dielectric ($\sigma = 0$). Then, regardless of the termination at $z = 0$, the equations to be solved are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu \frac{\partial \mathbf{H}}{\partial t} \tag{128a}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} \tag{128b}$$

For the geometry of the arrangements, $\mathbf{E} = E_x(z, t)\mathbf{a}_x$ and $\mathbf{H} = H_y(z, t)\mathbf{a}_y$, so that (128a) and (128b) simplify to

$$\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t} \tag{129a}$$

$$\frac{\partial H_y}{\partial z} = -\varepsilon \frac{\partial E_x}{\partial t} \tag{129b}$$

Combining the two equations by eliminating H_y, we obtain

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2} \tag{130}$$

which is the *wave equation*.

Wave Equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu \frac{\partial \mathbf{H}}{\partial t} \qquad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

For the one-dimensional case of

$\mathbf{E} = E_x(z,t)\,\mathbf{a}_x$ and $\mathbf{H} = H_y(z,t)\,\mathbf{a}_y$,

$$\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t} \qquad \frac{\partial H_y}{\partial z} = -\varepsilon \frac{\partial E_x}{\partial t}$$

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2}$$

One-dimensional wave equation

Slide Nos. 80-81

The wave equation has solutions of the form

$$E_x(z,t) = A \cos \omega\left(t - \sqrt{\mu\varepsilon}\, z + \phi^+\right) + B \cos \omega\left(t + \sqrt{\mu\varepsilon}\, z + \phi^-\right) \qquad (131)$$

The terms on the right side correspond to traveling waves propagating in the +*z* and –*z* directions, which we shall call the (+) and (–) waves, respectively, with the velocity $1/\sqrt{\mu\varepsilon}$, or $c/\sqrt{\mu_r\varepsilon_r}$, where $c = 1/\sqrt{\mu_0\varepsilon_0}$ is the velocity of light in free space. This can be seen by setting the derivative of the argument of the cosine function in each term equal to zero, or by plotting each term versus *z* for a few values of *t*, as on Slides 80 and 81, for the (+) and (–) waves, respectively. The corresponding solution for H_y is given by

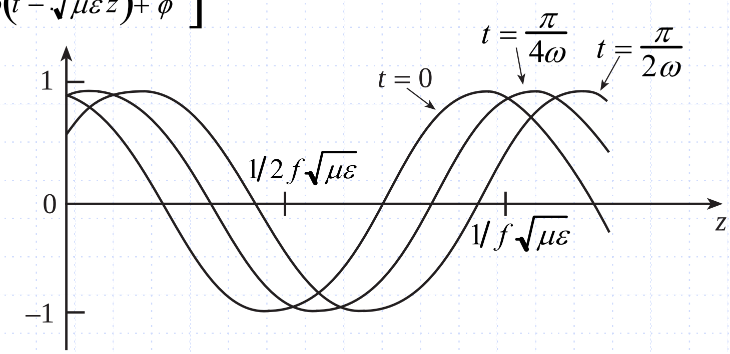
$$H_y(z,t) = \frac{1}{\sqrt{\mu/\varepsilon}}\left[A \cos \omega\left(t - \sqrt{\mu\varepsilon}\, z + \phi^+\right) - B \cos \omega\left(t + \sqrt{\mu\varepsilon}\, z + \phi^-\right)\right] \qquad (132)$$

Solution to the One-Dimensional Wave Equation

$$E_x(z,t) = A\cos\omega\left(t - \sqrt{\mu\varepsilon}\,z + \phi^+\right) + B\cos\omega\left(t + \sqrt{\mu\varepsilon}\,z + \phi^-\right)$$

Traveling wave propagating in the $+z$ direction

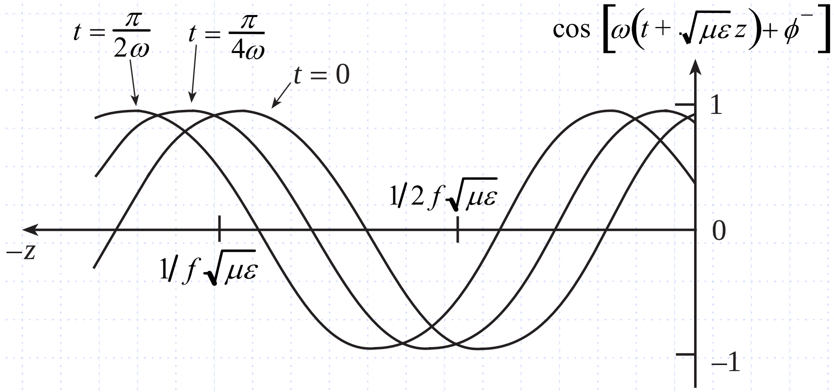
$$\cos\left[\omega\left(t - \sqrt{\mu\varepsilon}\,z\right) + \phi^+\right]$$



Solution to the One-Dimensional Wave Equation

$$H_y(z,t) = \frac{1}{\sqrt{\mu/\varepsilon}}\left[A\cos\omega\left(t - \sqrt{\mu\varepsilon}\,z + \phi^+\right) - B\cos\omega\left(t + \sqrt{\mu\varepsilon}\,z + \phi^-\right)\right]$$

Traveling wave propagating in the –*z* direction



Slide No. 82

For sinusoidal waves, which is the case at present, the velocity of propagation is known as the phase velocity, denoted by v_p, since it is the velocity with which a constant phase surface moves in the direction of propagation. The quantity $\omega\sqrt{\mu\varepsilon}$ is the magnitude of the rate of change of phase at a fixed time t, for either wave. It is known as the phase constant and is denoted by the symbol β. The quantity $\sqrt{\mu/\varepsilon}$, which is the ratio of the electric field intensity to the magnetic field intensity for the (+) wave, and the negative of such ratio for the (–) wave, is known as the intrinsic impedance of the medium. It is denoted by the symbol η. Thus, the phasor electric and magnetic fields can be written as

$$\bar{E}_x = \bar{A}e^{-j\beta z} + \bar{B}e^{j\beta z} \tag{133}$$

$$\bar{H}_y = \frac{1}{\eta}\left(\bar{A}e^{-j\beta z} - \bar{B}e^{j\beta z}\right) \tag{134}$$

General Solution in Phasor Form

$$\bar{E}_x = \bar{A} e^{-j\beta z} + \bar{B} e^{j\beta z}$$

$$\bar{H}_y = \frac{1}{\eta}\left(\bar{A} e^{-j\beta z} - \bar{B} e^{j\beta z}\right)$$

$$\bar{A} = A e^{j\phi^+}, \ \bar{B} = B e^{j\phi^-}$$

$$\beta = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{v_p},$$ Phase constant

$$\mathrm{v}_p = \frac{1}{\sqrt{\mu\varepsilon}},$$ Phase velocity

$$\eta = \sqrt{\frac{\mu}{\varepsilon}},$$ Intrinsic impedance

Slide Nos. 83-85

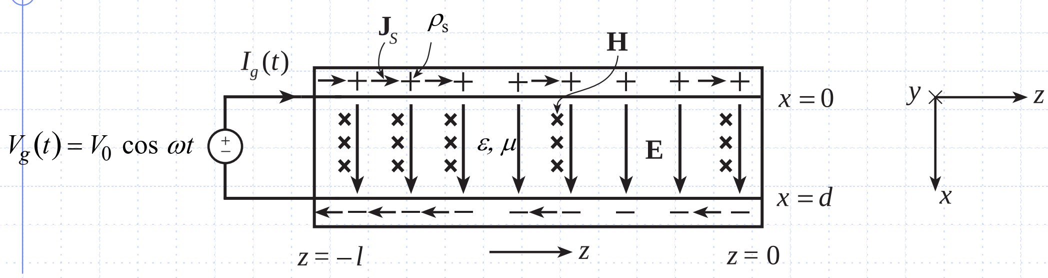
We may now use the boundary conditions for a given problem and obtain the specific solution for that problem. For the arrangement on Slide 64, the boundary conditions are $\bar{H}_y = 0$ at $z = 0$ and $\bar{E}_x = \bar{V}_g/d$ at $z = -l$. We thus obtain the particular solution for that arrangement to be

$$\bar{E}_x = \frac{\bar{V}_g}{d\ \cos\beta l}\cos\beta z \qquad (135)$$

$$\bar{H}_y = \frac{-j\bar{V}_g}{\eta d\ \cos\beta l}\sin\beta z \qquad (136)$$

which correspond to complete standing waves, resulting from the superposition of (+) and (–) waves of equal amplitude. Complete standing waves are characterized by pure half-sinusoidal variations for the amplitudes of the fields, as shown on Slide 84. For values of z at which the electric field amplitude is a maximum, the magnetic field amplitude is zero, and for values of z at which the electric field amplitude is zero, the magnetic field amplitude is a maximum. The fields are also out of phase in time, such that at any value of z, the magnetic field and the electric field differ in phase by $t = \pi/2\omega$.

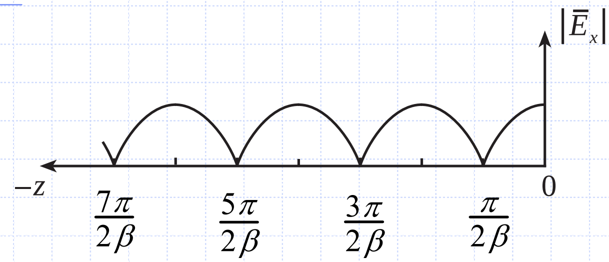
Example of Parallel-Plate Structure Open-Circuited at the Far End

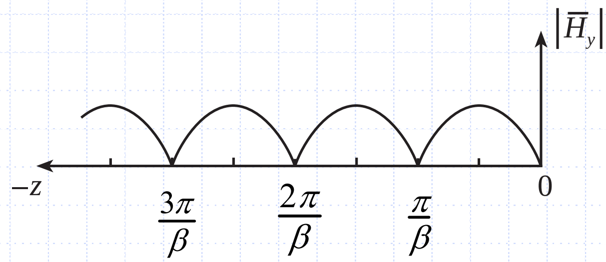


$$\left.\begin{aligned} \bar{H}_y &= 0 \text{ at } z = 0 \\ \bar{E}_x &= \frac{\bar{V}_g}{d} \text{ at } z = -l \end{aligned}\right\} \text{B.C.}$$

$$\bar{E}_x = \frac{\bar{V}_g}{d \cos \beta l} \cos \beta z \qquad \bar{H}_y = \frac{-j\bar{V}_g}{\eta d \cos \beta l} \sin \beta z$$

Standing Wave Patterns (Complete Standing Waves)





Complete Standing Waves

Complete standing waves are characterized by pure half-sinusoidal variations for the amplitudes of the fields. For values of z at which the electric field amplitude is a maximum, the magnetic field amplitude is zero, and for values of z at which the electric field amplitude is zero, the magnetic field amplitude is a maximum. The fields are also out of phase in time, such that at any value of z, the magnetic field and the electric field differ in phase by $t = \pi / 2\omega$.

Slide No. 86

Now, the current drawn from the voltage source is given by

$$\begin{aligned}\bar{I}_g &= w\left[\bar{H}_y\right]_{z=-l} \\ &= \frac{jw\bar{V}_g}{\eta d}\tan\beta l\end{aligned} \tag{137}$$

so that the input impedance of the structure is

$$\bar{Y}_{in} = \frac{\bar{I}_g}{\bar{V}_g} = j\frac{w}{\eta d}\tan\beta l \tag{138}$$

which can be expressed as a power series in βl. In particular, for $\beta l < \pi/2$,

$$\bar{Y}_{\text{in}} = j\frac{w}{\eta d}\left[\beta l + \frac{(\beta l)^3}{3} + \frac{2(\beta l)^5}{15} + \cdots\right] \tag{139}$$

The first term on the right side can be identified as belonging to the quasistatic approximation. Indeed for $\beta l << 1$, the higher order terms can be neglected, and

$$\begin{aligned}\bar{Y}_{\text{in}} &\approx \frac{jw}{\eta d}(\beta l) \\ &= j\omega\left(\frac{\varepsilon wl}{d}\right)\end{aligned} \tag{140}$$

same as that following from (108).

Input Admittance

$$\bar{I}_g = w\left[\bar{H}_y\right]_{z=-l} = \frac{jw\bar{V}_g}{\eta d}\tan\beta l$$

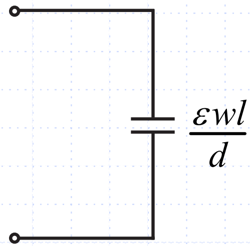
$$\bar{Y}_{\text{in}} = \frac{\bar{I}_g}{\bar{V}_g} = j\frac{w}{\eta d}\tan\beta l$$

$$\bar{Y}_{\text{in}} = j\frac{w}{\eta d}\left[\beta l + \frac{(\beta l)^3}{3} + \frac{2(\beta l)^5}{15} + \cdots\right]$$

For $\beta l << 1$,

$$\bar{Y}_{\text{in}} \approx \frac{jw}{\eta d}(\beta l)$$

$$= j\omega\left(\frac{\varepsilon w l}{d}\right)$$



Slide No. 87

It can now be seen that the condition $\beta l << 1$ dictates the range of validity for the quasistatic approximation for the input behavior of the structure. In terms of the frequency f of the source, this condition means that $f << v_p/2\pi l$, or in terms of the period $T = 1/f$, it means that $T >> 2\pi(l/v_p)$. Thus, as already mentioned, quasistatic fields are low-frequency approximations of time-varying fields that are complete solutions to Maxwell's equations, which represent wave propagation phenomena and can be approximated to the quasistatic character only when the times of interest are much greater than the propagation time, l/v_p, corresponding to the length of the structure. In terms of space variations of the fields at a fixed time, the wavelength $\lambda(= 2\pi/\beta)$, which is the distance between two consecutive points along the direction of propagation between which the phase difference is 2π, must be such that $l << \lambda/2\pi$; thus, the physical length of the structure must be a small fraction of the wavelength. In terms of amplitudes of the fields, what this means is that over the length of the structure, the field amplitudes are fractional portions of the first one-quarter sinusoidal variations at the $z = 0$ end in the figure on Slide 84, with the boundary conditions at the two ends of the structure always satisfied. Thus, because of the cos βz dependence of $\bar{E}_x$ on z, the electric field amplitude is essentially a constant, whereas because of the sin βz dependence of $\bar{H}_y$ on z, the magnetic field amplitude varies linearly with z. These are exactly the nature of the variations of the zero-order electric field and the first-order magnetic field, as discussed under electroquasistatic fields.

Condition for the Validity of the Quasistatic Approximation

The condition $\beta l \ll 1$ dictates the range of validity for the quasistatic approximation for the input behavior of the structure. In terms of the frequency f of the source, this condition means that $f \ll v_p/2\pi l$, or in terms of the period $T = 1/f$, it means that $T \gg 2\pi(l/v_p)$. Thus, quasistatic fields are low-frequency approximations of time-varying fields that are complete solutions to Maxwell's equations, which represent wave propagation phenomena and can be approximated to the quasistatic character only when the times of interest are much greater than the propagation time, l/v_p, corresponding to the length of the structure. In terms of space variations of the fields at a fixed time, the wavelength λ ($= 2\pi/\beta$), which is the distance between two consecutive points along the direction of propagation between which the phase difference is 2π, must be such that $l \ll \lambda/2\pi$; thus, the physical length of the structure must be a small fraction of the wavelength.

Slide No. 88

For frequencies slightly beyond the range of validity of the quasistatic approximation, we can include the second term in the infinite series on the right side of (139) and deduce the equivalent circuit in the following manner.

$$\begin{aligned}\bar{Y}_{\text{in}} &\approx j\frac{w}{\eta d}\left[\beta l + \frac{(\beta l)^3}{3}\right] \\ &= j\omega\left(\frac{\varepsilon wl}{d}\right)\left[1 + \left(\omega\frac{\varepsilon wl}{d}\right)\left(\omega\frac{\mu dl}{3w}\right)\right]\end{aligned} \tag{141}$$

or

$$\begin{aligned}\bar{Z}_{\text{in}} &= \frac{1}{j\omega\left(\frac{\varepsilon wl}{d}\right)\left[1 + \omega\left(\frac{\varepsilon wl}{d}\right)\left(\omega\frac{\mu dl}{3w}\right)\right]} \\ &\approx \frac{1}{j\omega(\varepsilon wl/d)} + j\omega\left(\frac{\mu dl}{3w}\right)\end{aligned} \tag{142}$$

Thus the input behavior is equivalent to that of a capacitor of value same as that for the quasistatic approximation in series with an inductor of value $\frac{1}{3}$ times the inductance found under the quasistatic approximation for the same arrangement but shorted at $z = 0$, by joining the two parallel plates. This series inductance is familiarly known as the "stray" inductance. But, all that has occurred is that the fractional portion of the sinusoidal variations of the field amplitudes over the length of the structure has increased, because the wavelength has decreased. As the frequency of the source is further increased, more and more terms in the infinite series need to be included and the equivalent circuit becomes more and more involved. But throughout all this range of frequencies, the overall input behavior is still capacitive, until a frequency is reached

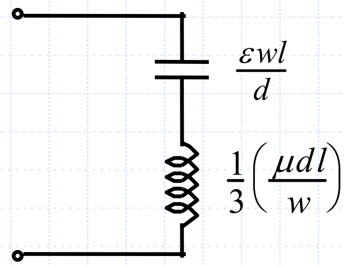
For frequencies slightly beyond the approximation $\beta l \ll 1$,

$$\bar{Y}_{\text{in}} \approx j\frac{w}{\eta d}\left[\beta l + \frac{(\beta l)^3}{3}\right]$$

$$= j\omega\left(\frac{\varepsilon w l}{d}\right)\left[1 + \left(\omega\frac{\varepsilon w l}{d}\right)\left(\omega\frac{\mu d l}{3w}\right)\right]$$

$$\bar{Z}_{\text{in}} = \frac{1}{j\omega\left(\frac{\varepsilon w l}{d}\right)\left[1 + \omega\left(\frac{\varepsilon w l}{d}\right)\left(\omega\frac{\mu d l}{3w}\right)\right]}$$

$$\approx \frac{1}{j\omega(\varepsilon w l/d)} + j\omega\left(\frac{\mu d l}{3w}\right)$$

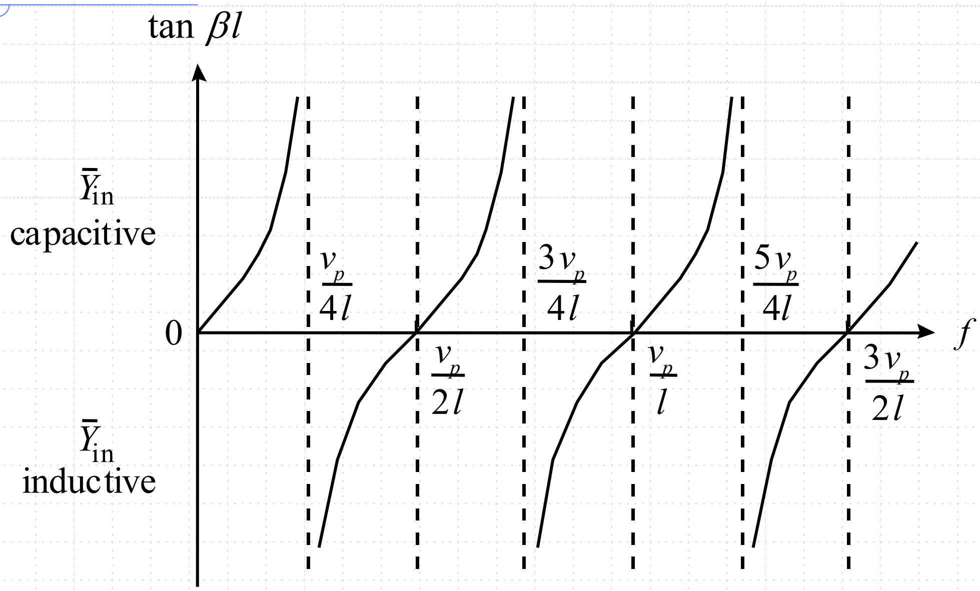


when βl crosses the value $\pi/2$, tan βl becomes negative and the input behavior changes to inductive!

Slide No. 89

In fact, a plot of tan βl versus f indicates that as the frequency is varied, the input behavior alternates between capacitive and inductive, an observation unpredictable without the complete solutions to Maxwell's equations. At the frequencies at which the input behavior changes from capacitive to inductive, the input admittance becomes infinity (short-circuit condition). The field amplitude variations along the length of the structure are then exactly odd integer multiples of one-quarter sinusoids. At the frequencies at which the input behavior changes from inductive to capacitive, the input admittance becomes zero (open-circuit condition). The field amplitude variations along the length of the structure are then exactly even integer multiple of one-quarter sinusoids, or integer multiples of one-half sinusoids.

In general,



tan βl
$\bar{Y}_{in}$
capacitive
$\bar{Y}_{in}$
inductive
0
f
$\frac{v_p}{4l}$
$\frac{v_p}{2l}$
$\frac{3v_p}{4l}$
$\frac{v_p}{l}$
$\frac{5v_p}{4l}$
$\frac{3v_p}{2l}$

Slide Nos. 90-91

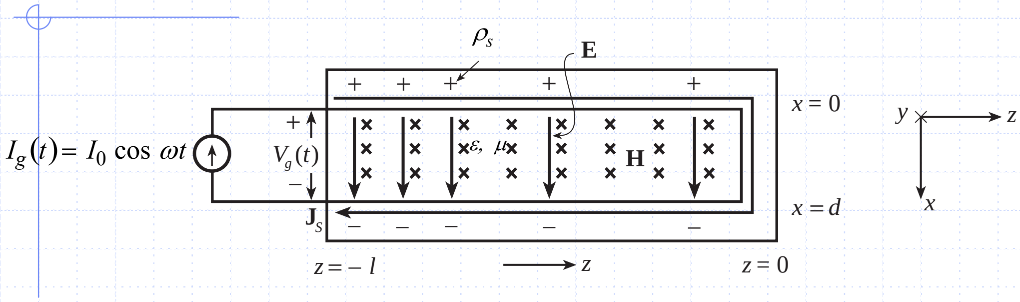
Turning now to the arrangement on Slide 68, the boundary conditions are $\bar{E}_x = 0$ at $z = 0$ and $\bar{H}_y = \bar{I}_g/w$ at $z = -l$. We thus obtain the particular solution for that arrangement to be

$$\bar{E}_x = -\frac{j\eta\bar{I}_g}{w\cos\beta l}\sin\beta z \tag{143}$$

$$\bar{H}_y = \frac{\bar{I}_g}{w\cos\beta l}\cos\beta z \tag{144}$$

which, once again, correspond to complete standing waves, resulting from the superposition of (+) and (–) waves of equal amplitude, and characterized by pure half-sinusoidal variations for the amplitudes of the fields, as shown on Slide 91, which are of the same nature as on Slide 84, except that the variations for the electric and magnetic fields are interchanged.

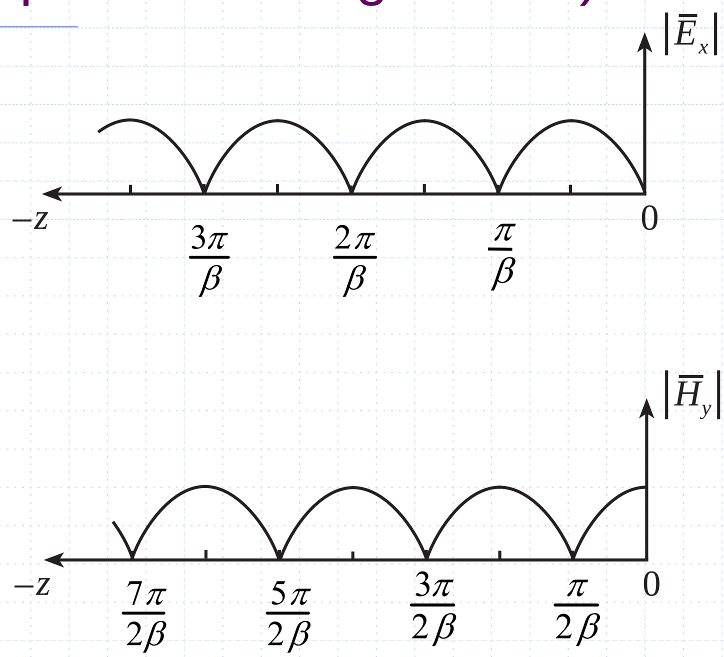
Example of Parallel-Plate Structure Short-Circuited at the Far End



$$\left.\begin{aligned} \bar{E}_x &= 0 \text{ at } z = 0 \\ \bar{H}_y &= \frac{\bar{I}_g}{w} \text{ at } z = -l \end{aligned}\right\} \text{B.C.}$$

$$\bar{E}_x = -\frac{j\eta \bar{I}_g}{w \cos \beta l} \sin \beta z \qquad \bar{H}_y = \frac{\bar{I}_g}{w \cos \beta l} \cos \beta z$$

Standing Wave Patterns
(Complete Standing Waves)



Slide No. 92

Now, the voltage across the current source is given by

$$\begin{aligned}\bar{V}_g &= d\left[\bar{E}_x\right]_{z=-l} \\ &= \frac{j\eta d \bar{I}_g}{w}\tan\beta l\end{aligned} \tag{145}$$

so that the input impedance of the structure is

$$\bar{Z}_{\text{in}} = \frac{\bar{V}_g}{\bar{I}_g} = j\frac{\eta d}{w}\tan\beta l \tag{146}$$

which can be expressed as a power series in βl. In particular, for $\beta l < \pi/2$,

$$\bar{Z}_{\text{in}} = j\frac{\eta d}{w}\left[\beta l + \frac{(\beta l)^3}{3} + \frac{2(\beta l)^5}{15} + \cdots\right] \tag{147}$$

Once again, the first term on the right side can be identified as belonging to the quasistatic approximation. Indeed for $\beta l << 1$,

$$\begin{aligned}\bar{Z}_{\text{in}} &\approx \frac{\eta d}{w}(\beta l) \\ &= j\omega\left(\frac{\mu dl}{w}\right)\end{aligned} \tag{148}$$

same as that following from (114), and all the discussion pertinent to the condition for the validity of the quasistatic approximation for the structure on Slide 64 applies also to the structure on Slide 68, with the roles of the electric and magnetic fields interchanged. For $l << \lambda/2\pi$, the field amplitudes over the length of the structure are fractional portions of the first one-quarter sinusoidal variations at the $z = 0$ end in the figure on Slide 91, with the boundary conditions at the two ends always satisfied. Thus, because of the cos βz dependence of $\bar{H}_y$ on z, the magnetic field amplitude is essentially a constant, whereas because of the sin βz dependence of $\bar{E}_x$ on z, the electric field amplitude varies linearly with z. These are exactly the nature of the variations of the zero-order magnetic field and the first-order electric field, as discussed under magnetoquasistatic fields.

Input Impedance

$$\bar{V}_g = d\left[\bar{E}_x\right]_{z=-l} = \frac{j\eta d\bar{I}_g}{w}\tan\beta l$$

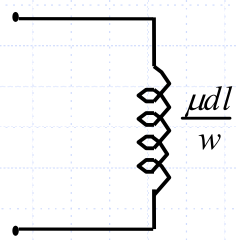
$$\bar{Z}_{\text{in}} = \frac{\bar{V}_g}{\bar{I}_g} = j\frac{\eta d}{w}\tan\beta l$$

$$\bar{Z}_{\text{in}} = j\frac{\eta d}{w}\left[\beta l + \frac{(\beta l)^3}{3} + \frac{2(\beta l)^5}{15} + \cdots\right]$$

For $\beta l << 1$

$$\bar{Z}_{\text{in}} \approx \frac{\eta d}{w}(\beta l)$$

$$= j\omega\left(\frac{\mu d l}{w}\right)$$



Slide No. 93

For frequencies slightly beyond the range of validity of the quasistatic approximation, we can include the second term in the infinite series on the right side of (147) and deduce the equivalent circuit in the following manner.

$$\begin{aligned}\bar{Z}_{\text{in}} &\approx \frac{j\eta d}{w}\left[\beta l + \frac{(\beta l)^3}{3}\right] \\ &= j\omega\left(\frac{\mu dl}{w}\right)\left[1+\left(\omega\frac{\mu dl}{w}\right)\left(\omega\frac{\varepsilon wl}{3d}\right)\right]\end{aligned} \tag{149}$$

or

$$\begin{aligned}\bar{Y}_{\text{in}} &= \frac{1}{j\omega\left(\frac{\mu dl}{w}\right)\left[1+\left(\omega\frac{\mu dl}{w}\right)\left(\omega\frac{\varepsilon wl}{3d}\right)\right]} \\ &\approx \frac{1}{j\omega(\mu dl/w)} + j\omega\left(\frac{\varepsilon wl}{3d}\right)\end{aligned} \tag{150}$$

Thus the input behavior is equivalent to that of an inductor of value same as that for the quasistatic approximation in parallel with a capacitor of value $\frac{1}{3}$ times the capacitance found under the quasistatic approximation for the same arrangement but open at $z = 0$, without the two plates joined. This parallel capacitance is familiarly known as the "stray" capacitance. But again, all that has occurred is that the fractional portion of the sinusoidal variations of the field amplitudes over the length of the structure has increased, because the wavelength has decreased. As the frequency of the source is further increased, more and more terms in the infinite series need to be included and the equivalent circuit becomes more and more involved. But throughout all this range of

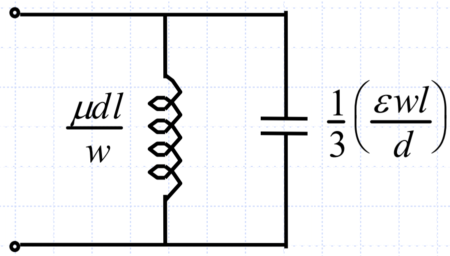
For frequencies slightly beyond the approximation $\beta l << 1$,

$$\bar{Z}_{\text{in}} \approx \frac{j\eta d}{w}\left[\beta l + \frac{(\beta l)^3}{3}\right]$$

$$= j\omega\left(\frac{\mu d l}{w}\right)\left[1 + \left(\omega \frac{\mu d l}{w}\right)\left(\omega \frac{\varepsilon w l}{3d}\right)\right]$$

$$\bar{Y}_{\text{in}} = \frac{1}{j\omega\left(\frac{\mu d l}{w}\right)\left[1 + \left(\omega \frac{\mu d l}{w}\right)\left(\omega \frac{\varepsilon w l}{3d}\right)\right]}$$

$$\approx \frac{1}{j\omega(\mu d l / w)} + j\omega\left(\frac{\varepsilon w l}{3d}\right)$$

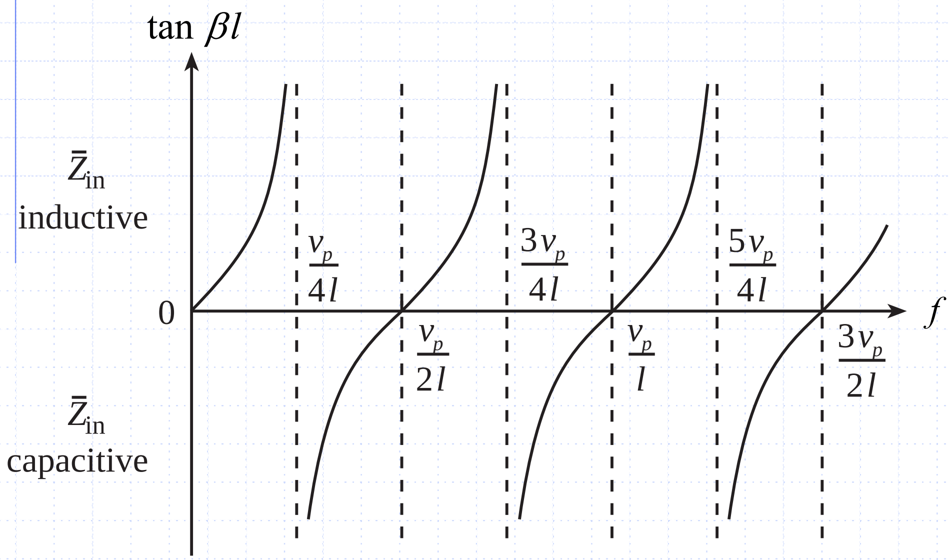


frequencies, the overall input behavior is still inductive, until a frequency is reached when βl crosses the value $\pi/2$, tan βl becomes negative and the input behavior changes to capacitive.

Slide No. 94

In fact, the plot of tan βl versus f indicates that as the frequency is varied, the input behavior alternates between inductive and capacitive, an observation unpredictable without the complete solutions to Maxwell's equations. At the frequencies at which the input behavior changes from inductive to capacitive, the input impedance becomes infinity (open-circuit condition). The field amplitude variations along the length of the structure are then exactly odd integer multiples of one-quarter sinusoids. At the frequencies at which the input behavior changes from capacitive to inductive, the input impedance becomes zero (short-circuit condition). The field amplitude variations along the length of the structure are then exactly even integer multiples of one-quarter sinusoids, or integer multiples of one-half sinusoids.

In general,


tan βl
Z̄in
inductive
0
Z̄in
capacitive
vp/4l
vp/2l
3vp/4l
vp/l
5vp/4l
3vp/2l
f

Slide No. 95

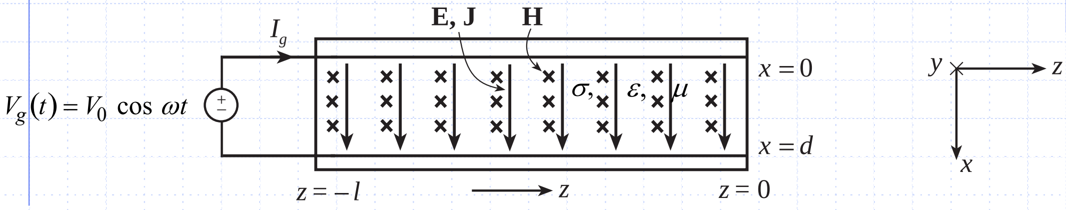
<u>The Distributed Circuit Concept</u>

We have seen that, from the circuit point of view, the structure on Slide 50 behaves like a capacitor for the static case and the capacitive character is essentially retained for its input behavior for sinusoidally time-varying excitation at frequencies low enough to be within the range of validity of the quasistatic approximation. Likewise, we have seen that, from a circuit point of view, the structure on Slide 54 behaves like an inductor for the static case and the inductive character is essentially retained for its input behavior for sinusoidally time-varying excitation at frequencies low enough to be within the range of validity of the quasistatic approximation. For both structures, at an arbitrarily high enough frequency, the input behavior can be obtained only by obtaining complete (wave) solutions to Maxwell's equations, subject to the appropriate boundary conditions. The question to ask then is: Is there a circuit equivalent for the structure itself, independent of the termination, that is representative of the phenomenon taking place along the structure and valid at any arbitrary frequency, to the extent that the material parameters themselves are independent of frequency? The answer is, yes, under certain conditions, giving rise to the concept of the *distributed circuit*.

To develop and discuss the concept of the distributed circuit using a more general case than that allowed by the arrangements on Slides 50 and 54, let us consider the case of the structure on Slide 58 driven by a sinusoidally time-varying source, as in figure (a) on Slide 72. Then the equations to be solved are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu \frac{\partial \mathbf{H}}{\partial t} \tag{151a}$$

Distributed Circuit Concept



$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t} = \sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathrm{J}_c + \frac{\partial \mathbf{D}}{\partial t} = \sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \qquad (151b)$$

Slide No. 96

For the geometry of the arrangement, $\mathbf{E} = E_x(z, t)\mathbf{a}_x$ and $\mathbf{H} = H_y(z, t)\mathbf{a}_y$, so that (151a) and (151b) simplify to

$$\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t} \qquad (152a)$$

$$\frac{\partial H_y}{\partial z} = -\sigma E_x - \varepsilon \frac{\partial E_x}{\partial t} \qquad (152b)$$

Now, since E_z and H_z are zero, we can, in a given z = constant plane, uniquely define a voltage between the plates in terms of the electric field intensity in that plane, and a current crossing that plane in one direction on the top plate and in the opposite direction on the bottom plate in terms of the magnetic field intensity in that plane. These are given by

$$V(z, t) = dE_x(z, t) \qquad (153a)$$

$$I(z, t) = wH_y(z, t) \qquad (153b)$$

Parallel-Plate Structure

For $\mathbf{E} = E_x(z,t)\,\mathbf{a}_x$ and $\mathbf{H} = H_y(z,t)\,\mathbf{a}_y$,

$$\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t} \qquad \frac{\partial H_y}{\partial z} = -\sigma E_x - \varepsilon \frac{\partial E_x}{\partial t}$$

Since $E_z = 0$ and $H_z = 0$, we can uniquely define

$V(z,t) = d\,E_x(z,t)$ — Voltage between the plates in a given z = constant plane

$I(z,t) = wH_y(z,t)$ — Current crossing a given z = constant plane

Slide No. 97

Substituting (153a) and (153b) in (152a) and (152b), and rearranging, we obtain

$$\frac{\partial V(z,t)}{\partial z} = -\left[\frac{\mu d}{w}\right] \frac{\partial I(z,t)}{\partial t} \tag{154a}$$

$$\frac{\partial I(z,t)}{\partial z} = -\left[\frac{\sigma w}{d}\right] V(z,t) - \left[\frac{\varepsilon w}{d}\right] \frac{\partial V(z,t)}{\partial t} \tag{154b}$$

Writing the derivates with respect to z on the left sides of the equations in terms of limits as $\Delta z \rightarrow 0$, and multiplying by Δz on both sides of the equations provides the equivalent circuit for a section of length Δz of the structure, in which the quantities $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$, given by

$$\mathcal{L} = \frac{\mu d}{w} \tag{155a}$$

$$\mathcal{C} = \frac{\varepsilon w}{d} \tag{155b}$$

$$\mathcal{G} = \frac{\sigma w}{d} \tag{155c}$$

are the inductance per unit length, capacitance per unit length, and conductance per unit length, respectively, of the structure, all computed from static field analysis, except that now they are expressed in terms of "per unit length," and not for the entire structure in a "lump."

Circuit Equivalent

Then

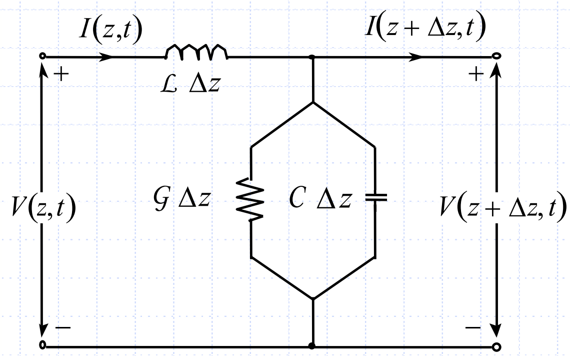
$$\frac{\partial V(z,t)}{\partial z} = -\left(\frac{\mu d}{w}\right)\frac{\partial I(z,t)}{\partial t}$$

$$\frac{\partial I(z,t)}{\partial z} = -\left(\frac{\sigma w}{d}\right)V(z,t) - \left(\frac{\varepsilon w}{d}\right)\frac{\partial V(z,t)}{\partial t}$$

$$\mathcal{L} = \frac{\mu d}{w}$$

$$\mathcal{C} = \frac{\varepsilon w}{d}$$

$$\mathcal{G} = \frac{\sigma w}{d}$$



Slide No. 98

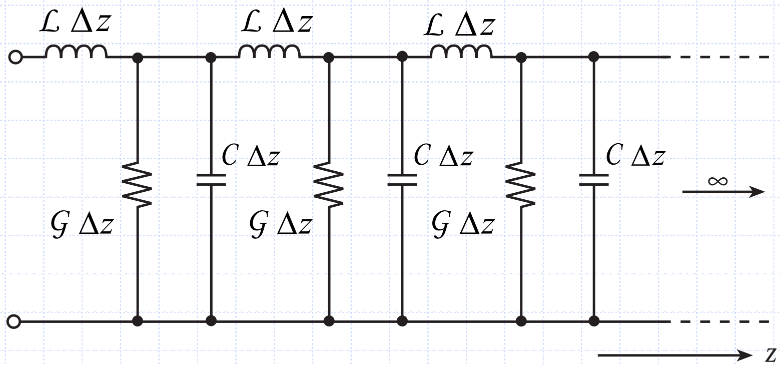
Transmission Line

It then follows that the circuit representation of the entire structure consists of an infinite number of such sections in cascade. Such a circuit is known as a *distributed circuit*. The distributed circuit notion arises from the fact that the inductance, capacitance, and conductance are distributed uniformly and overlappingly along the structure. A physical interpretation of the distributed-circuit concept follows from energy considerations, based on the properties that inductance, capacitance, and conductance are elements associated with energy storage in the magnetic field, energy storage in the electric field, and power dissipation due to conduction current flow, in the material. Since these phenomena occur continuously and overlappingly along the structure, the inductance, capacitance, and conductance must be distributed uniformly and overlappingly along the structure.

A physical structure for which the distributed circuit concept is applicable is familiarly known as a *transmission line*. The parallel-plate arrangement on Slides 50, 54, and 58 is a special case of a transmission line, known as the *parallel-plate* line, in which the waves are called *uniform plane waves*, since the fields are uniform in the z = constant planes. In general, a transmission line consists of two parallel conductors having arbitrary cross-sections and the waves are transverse electromagnetic, or TEM, waves, for which the fields are nonuniform in the z = constant planes but satisfying the property of both electric and magnetic fields having no components along the direction of propagation, that is, parallel to the conductors.

All transmission lines having perfect conductors are governed by the equation

Distributed Circuit Representation; Transmission Line



$$\frac{\partial V(z,t)}{\partial z} = -\mathcal{L}\frac{\partial I(z,t)}{\partial t}$$

$$\mathcal{L}\mathcal{C} = \mu\varepsilon$$

$$\frac{\partial I(z,t)}{\partial z} = -\mathcal{G}V(z,t) - \mathcal{C}\frac{\partial V(z,t)}{\partial t}$$

$$\frac{\mathcal{G}}{\mathcal{C}} = \frac{\sigma}{\varepsilon}$$

$$\frac{\partial V(z,t)}{\partial z} = -\mathcal{L}\,\frac{\partial I(z,t)}{\partial t} \tag{156a}$$

$$\frac{\partial I(z,t)}{\partial z} = -\mathcal{G}V(z,t) - \mathcal{C}\,\frac{\partial V(z,t)}{\partial t} \tag{156b}$$

which are known as the transmission-line equations. The values of $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ differ from one line to another, and depend on the cross-sectional geometry of the conductors. For the parallel-plate line, $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$ are given by (155a), (155b), and (155c), respectively. Note that

$$\mathcal{L}\mathcal{C} = \mu\varepsilon \tag{157a}$$

$$\frac{\mathcal{G}}{\mathcal{C}} = \frac{\sigma}{\varepsilon} \tag{157b}$$

a set of relations, which is applicable to any line governed by (156a) and (156b). Thus for a given set of material parameters, only one of the three parameters, $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{G}$, is independent.

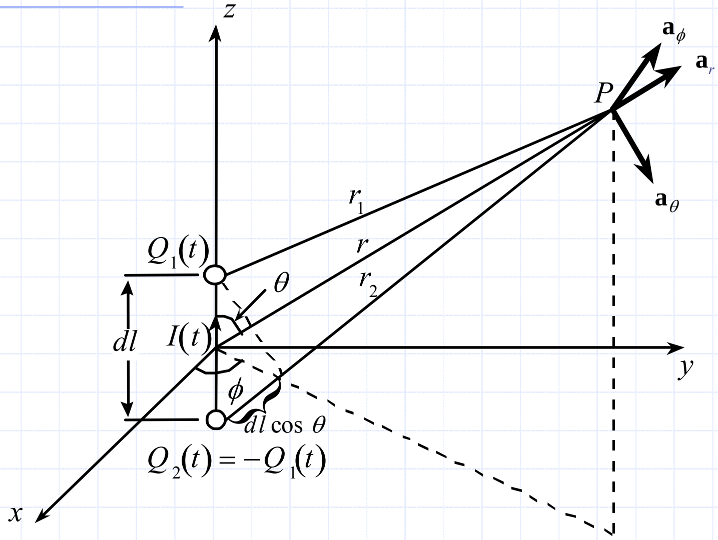
In practice, the conductors are imperfect, adding a resistance per unit length and additional inductance per unit length in the series branches of the distributed circuit. Although the waves are then no longer exactly TEM waves, the distributed circuit representation is commonly used for transmission lines with imperfect conductors. Another consideration that arises in practice is that the material parameters and hence the line parameters can be functions of frequency.

Slide No. 99

The Hertzian Dipole

We have seen the development of solutions to Maxwell's equations, beginning with static fields and spanning the frequency domain from quasistatic approximations at

Hertzian Dipole



$$I(t) = I_0 \cos \omega t$$

$$Q_1(t) = \frac{I_0}{\omega} \sin \omega t$$

$$Q_2(t) = -\frac{I_0}{\omega} \sin \omega t$$

$$= -Q_1(t)$$

low frequencies to waves for beyond quasistatics. Finally, we shall now develop the solution for the electromagnetic field due to a Hertzian dipole by making use of the thread of statics-quasistatics-waves, as compared to the commonly used approach based on the magnetic vector potential, for a culminating experience of revisiting the fundamentals of engineering electromagnetics.

The Hertzian dipole is an elemental antenna consisting of an infinitesimally long piece of wire carrying an alternating current $I(t)$. To maintain the current flow in the wire, we postulate two point charges $Q_1(t)$ and $Q_2(t)$ terminating the wire at its two ends, so that the law of conservation of charge is satisfied. Thus, if

$$I(t) = I_0 \cos \omega t \tag{158}$$

then

$$Q_1(t) = \frac{I_0}{\omega} \sin \omega t \tag{159a}$$

$$Q_2(t) = -\frac{I_0}{\omega} \sin \omega t = -Q_1(t) \tag{159b}$$

Slide No. 100

For $d/dt = 0$, the charges are static and the current is zero. The field is simply the electrostatic field due to the electric dipole made up of $Q_1 = -Q_2 = Q_0$. Applying (70) to the geometry in the figure on Slide 99, we write the electrostatic potential at the point P due to the dipole located at the origin to be

$$\Phi = \frac{Q_0}{4\pi\varepsilon}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \tag{160}$$

In the limit $dl \to 0$, keeping the dipole moment $Q_0(dl)$ fixed, we get

Derivation of Hertzian Dipole Fields

For $d/dt = 0$, $Q_1 = -Q_2 = Q_0$, $I = 0$

$$\Phi = \frac{Q_0}{4\pi\varepsilon}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

In the limit $dl \to 0$, keeping $Q_0\,(dl)$ constant,

$$\Phi = \frac{Q_0(dl)\cos\theta}{4\pi\varepsilon r^2}$$

$$\mathbf{E} = -\nabla\Phi = \frac{Q_0(dl)}{4\pi\varepsilon r^3}\left(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta\right)$$

$$\Phi = \frac{Q_0 (dl) \cos \theta}{4\pi\varepsilon r^2} \tag{161}$$

so that the electrostatic field at the point *P* due to the dipole is given by

$$\mathbf{E} = -\nabla\Phi = \frac{Q_0 (dl)}{4\pi\varepsilon r^3} \left(2 \cos \theta \, \mathbf{a}_r + \sin \theta \, \mathbf{a}_\theta\right) \tag{162}$$

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With time variations in the manner $Q_1(t) = -Q_2(t) = Q_0 \sin \omega t$, so that $I_0 = \omega Q_0$, and at low frequencies, the situation changes to electroquasistatic with the electric field of amplitude proportional to the zeroth power in ω given by

$$\mathbf{E}_0 = \frac{Q_0 (dl) \sin \omega t}{4\pi\varepsilon r^3} \left(2 \cos \theta \, \mathbf{a}_r + \sin \theta \, \mathbf{a}_\theta\right) \tag{163}$$

The corresponding magnetic field of amplitude proportional to the first power in ω is given by the solution of

$$\nabla \times \mathbf{H}_1 = \frac{\partial \mathbf{D}_0}{\partial t} = \varepsilon \frac{\partial \mathbf{E}_0}{\partial t} \tag{164}$$

For the geometry associated with the arrangement, this reduces to

$$\begin{vmatrix} \dfrac{\mathbf{a}_r}{r^2 \sin \theta} & \dfrac{\mathbf{a}_\theta}{r \sin \theta} & \dfrac{\mathbf{a}_\phi}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & 0 \\ 0 & 0 & r \sin \theta \, H_{\phi 1} \end{vmatrix} = \varepsilon \frac{\partial \mathbf{E}_0}{\partial t} \tag{165}$$

so that

$$\mathbf{H}_1 = \frac{\omega Q_0 (dl) \cos \omega t}{4\pi r^2} \sin \theta \, \mathbf{a}_\phi \tag{166}$$

Derivation of Hertzian Dipole Fields

For $Q_1(t) = -Q_2(t) = Q_0 \sin \omega t$, $I_0 = \omega Q_0$, and at low frequencies,

$$\mathbf{E}_0 = \frac{Q_0 (dl) \sin \omega t}{4\pi\varepsilon r^3} \left(2 \cos\theta\, \mathbf{a}_r + \sin\theta\, \mathbf{a}_\theta\right)$$

$$\nabla \times \mathbf{H}_1 = \frac{\partial \mathbf{D}_0}{\partial t} = \varepsilon \frac{\partial \mathbf{E}_0}{\partial t} \qquad \begin{vmatrix} \dfrac{\mathbf{a}_r}{r^2 \sin\theta} & \dfrac{\mathbf{a}_\theta}{r \sin\theta} & \dfrac{\mathbf{a}_\phi}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & 0 \\ 0 & 0 & r \sin\theta\, H_{\phi 1} \end{vmatrix} = \varepsilon \frac{\partial \mathbf{E}_0}{\partial t}$$

$$\mathbf{H}_1 = \frac{\omega Q_0 (dl) \cos \omega t}{4\pi r^2} \sin\theta\, \mathbf{a}_\phi$$

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To extend the solutions for the fields for frequencies beyond the range of validity of the quasistatic approximation, we recognize that the situation then corresponds to wave propagation. With the dipole at the origin, the waves propagate radially away from it so that the time functions sin ωt and cos ωt in (163) and (166) need to be replaced by sin $(\omega t - \beta r)$ and cos $(\omega t - \beta r)$, respectively, where $\beta = \omega\sqrt{\mu\varepsilon}$ is the phase constant. Therefore, let us on this basis alone and without any other considerations, write the field expressions as

$$\mathbf{E} = \frac{I_0(dl)\sin(\omega t - \beta r)}{4\pi\varepsilon\omega r^3}\left(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta\right) \tag{167}$$

$$\mathbf{H} = \frac{I_0(dl)\cos(\omega t - \beta r)}{4\pi r^2}\sin\theta\,\mathbf{a}_\phi \tag{168}$$

where we have also replaced Q_0 by I_0/ω, and pose the question as to whether or not these expressions represent the solution for the electromagnetic field due to the Hertzian dipole. The answer is "no," since they do not satisfy Maxwell's curl equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu\frac{\partial \mathbf{H}}{\partial t} \tag{169a}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} = \varepsilon\frac{\partial \mathbf{E}}{\partial t} \tag{169b}$$

which can be verified by substituting them into the equations.

Derivation of Hertzian Dipole Fields

$$\mathbf{E} = \frac{I_0 (dl) \sin(\omega t - \beta r)}{4\pi \varepsilon \omega r^3} \left(2 \cos\theta \, \mathbf{a}_r + \sin\theta \, \mathbf{a}_\theta\right)$$

$$\mathbf{H} = \frac{I_0 (dl) \cos(\omega t - \beta r)}{4\pi r^2} \sin\theta \, \mathbf{a}_\phi$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

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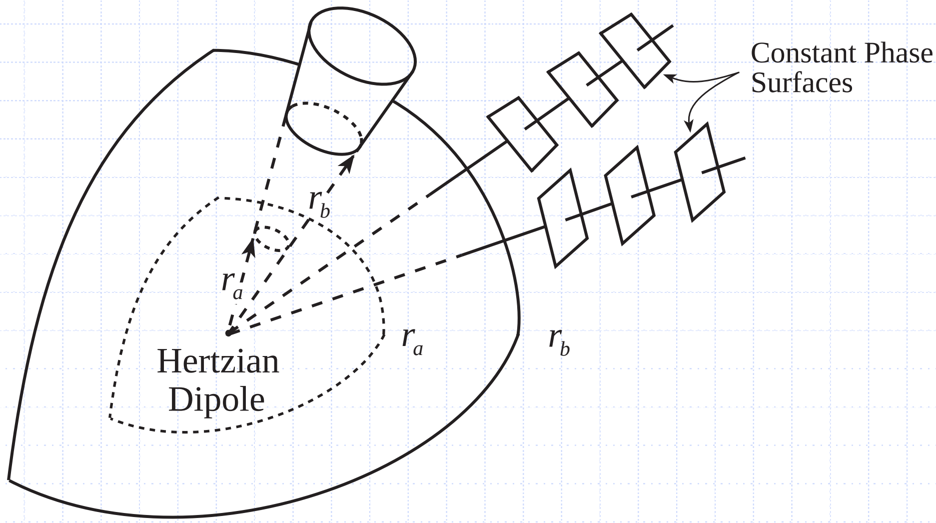
There is more than one way of resolving this discrepancy, but we shall here do it from physical considerations. Even a cursory look at the solutions for the fields given by (167) and (168) points to the problem, since the Poynting vector **E x H** corresponding to them is proportional to $1/r^5$, and there is no real power flow associated with them because they are out of phase in ωt by $\pi/2$. But, we should expect that the fields contain terms proportional to $1/r$, which are in phase, from considerations of real power flow in the radial direction and from the behavior of the waves viewed locally over plane areas normal to the radial lines emanating from the Hertzian dipole, and electrically far from it ($\beta r >> 1$), to be approximately that of uniform plane waves with the planes as their constant phase surfaces. To elaborate upon this, let us consider two spherical surfaces of radii r_a and r_b and centered at the dipole and insert a cone through these two surfaces such that its vertex is at the antenna, as shown in the figure. Then the power crossing the portion of the spherical surface of radius r_b inside the cone must be the same as the power crossing the spherical surface of radius r_a inside the cone. Since these surface areas are proportional to the square of the radius and since the surface integral of the Poynting vector gives the power, the Poynting vector must have an r-component proportional to $1/r^2$, and it follows that the solutions for E_θ and H_ϕ must contain terms proportional to $1/r$ and in phase.

Derivation of Hertzian Dipole Fields



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Thus let us modify the expression for **H** given by (168) by adding a second term containing $1/r$ in the manner

$$\mathbf{H} = \frac{I_0(dl)\sin\theta}{4\pi}\left[\frac{\cos(\omega t - \beta r)}{r^2} + \frac{A\cos(\omega t - \beta r + \delta)}{r}\right]\mathbf{a}_\phi \qquad (170)$$

where A and δ are constants to be determined. Then, from Maxwell's curl equation for **H**, given by (169b), we obtain

$$\begin{aligned}\mathbf{E} = {} & \frac{2I_0(dl)\cos\theta}{4\pi\varepsilon\omega}\left[\frac{\sin(\omega t - \beta r)}{r^3} + \frac{A\sin(\omega t - \beta r + \delta)}{r^2}\right]\mathbf{a}_r \\ & + \frac{I_0(dl)\sin\theta}{4\pi\varepsilon\omega}\left[\frac{\sin(\omega t - \beta r)}{r^3} + \frac{\beta\sin(\omega t - \beta r)}{r^2}\right. \\ & \left. + \frac{A\beta\cos(\omega t - \beta r + \delta)}{r}\right]\mathbf{a}_\theta\end{aligned} \qquad (171)$$

Derivation of Hertzian Dipole Fields

Modify **H** as follows:

$$\mathbf{H} = \frac{I_0(dl)\sin\theta}{4\pi}\left[\frac{\cos(\omega t - \beta r)}{r^2} + \frac{A\cos(\omega t - \beta r + \delta)}{r}\right]\mathbf{a}_\phi$$

Then, from Maxwell's curl equation for **H**,

$$\mathbf{E} = \frac{2I_0(dl)\cos\theta}{4\pi\varepsilon\omega}\left[\frac{\sin(\omega t - \beta r)}{r^3} + \frac{A\sin(\omega t - \beta r + \delta)}{r^2}\right]\mathbf{a}_r$$
$$+ \frac{I_0(dl)\sin\theta}{4\pi\varepsilon\omega}\left[\frac{\sin(\omega t - \beta r)}{r^3} + \frac{\beta\sin(\omega t - \beta r)}{r^2}\right.$$
$$\left. + \frac{A\beta\cos(\omega t - \beta r + \delta)}{r}\right]\mathbf{a}_\theta$$

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Now, substituting this in Maxwell's curl equation for **E** given by (169a), we get

$$\mathbf{H} = \frac{I_0(dl)\sin\theta}{4\pi}\left[\frac{2\sin(\omega t-\beta r)}{\beta r^3} + \frac{2A\cos(\omega t-\beta r+\delta)}{\beta^2 r^3} + \frac{\cos(\omega t-\beta r)}{r^2} + \frac{A\cos(\omega t-\beta r+\delta)}{r}\right]\mathbf{a}_\phi \tag{172}$$

But (172) must be the same as (170). Therefore, we set

$$\frac{2\sin(\omega t-\beta r)}{\beta r^3} + \frac{2A\cos(\omega t-\beta r+\delta)}{\beta^2 r^3} = 0 \tag{173}$$

which gives us

$$\delta = \frac{\pi}{2} \tag{174}$$

$$A = \beta \tag{175}$$

Derivation of Hertzian Dipole Fields

Substituting in Maxwell's curl equation for **E**, we get

$$\mathbf{H} = \frac{I_0(dl)\sin\theta}{4\pi}\left[\frac{2\sin(\omega t-\beta r)}{\beta r^3} + \frac{2A\cos(\omega t-\beta r+\delta)}{\beta^2 r^3} + \frac{\cos(\omega t-\beta r)}{r^2} + \frac{A\cos(\omega t-\beta r+\delta)}{r}\right]\mathbf{a}_\phi$$

For the two expressions for **H** to be consistent,

$$\frac{2\sin(\omega t-\beta r)}{\beta r^3} + \frac{2A\cos(\omega t-\beta r+\delta)}{\beta^2 r^3} = 0$$

$$\delta = \frac{\pi}{2} \qquad A = \beta$$

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Substituting (174) and (175) in (171) and (172), we then have the complete electromagnetic field due to the Hertzian dipole given by

$$\mathbf{E} = \frac{2I_0(dl)\cos\theta}{4\pi\varepsilon\omega}\left[\frac{\sin(\omega t - \beta r)}{r^3} + \frac{\beta\cos(\omega t - \beta r)}{r^2}\right]\mathbf{a}_r + \frac{I_0(dl)\sin\theta}{4\pi\varepsilon\omega}\left[\frac{\sin(\omega t - \beta r)}{r^3} + \frac{\beta\cos(\omega t - \beta r)}{r^2} - \frac{\beta^2\sin(\omega t - \beta r)}{r}\right]\mathbf{a}_\theta \tag{176}$$

$$\mathbf{H} = \frac{I_0(dl)\sin\theta}{4\pi}\left[\frac{\cos(\omega t - \beta r)}{r^2} - \frac{\beta\sin(\omega t - \beta r)}{r}\right]\mathbf{a}_\phi \tag{177}$$

Hertzian Dipole Fields

$$\mathbf{E} = \frac{2I_0(dl)\cos\theta}{4\pi\varepsilon\omega}\left[\frac{\sin(\omega t - \beta r)}{r^3} + \frac{\beta\cos(\omega t - \beta r)}{r^2}\right]\mathbf{a}_r$$

$$+ \frac{I_0(dl)\sin\theta}{4\pi\varepsilon\omega}\left[\frac{\sin(\omega t - \beta r)}{r^3} + \frac{\beta\cos(\omega t - \beta r)}{r^2} - \frac{\beta^2\sin(\omega t - \beta r)}{r}\right]\mathbf{a}_\theta$$

$$\mathbf{H} = \frac{I_0(dl)\sin\theta}{4\pi}\left[\frac{\cos(\omega t - \beta r)}{r^2} - \frac{\beta\sin(\omega t - \beta r)}{r}\right]\mathbf{a}_\phi$$

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Expressed in phasor form and with some rearrangement, the field components are given by

$$\overline{E}_r = \frac{2\beta^2 \eta I_0 (dl) \cos\theta}{4\pi} \left[-j\frac{1}{(\beta r)^3} + \frac{1}{(\beta r)^2} \right] e^{-j\beta r} \tag{178}$$

$$\overline{E}_\theta = \frac{\beta^2 \eta I_0 (dl) \sin\theta}{4\pi} \left[-j\frac{1}{(\beta r)^3} + \frac{1}{(\beta r)^2} + j\frac{1}{\beta r} \right] e^{-j\beta r} \tag{179}$$

$$\overline{H}_\phi = \frac{\beta^2 I_0 (dl) \sin\theta}{4\pi} \left[\frac{1}{(\beta r)^2} + j\frac{1}{\beta r} \right] e^{-j\beta r} \tag{180}$$

Hertzian Dipole Fields

$$\overline{E}_r = \frac{2\beta^2 \eta I_0 (dl) \cos\theta}{4\pi}\left[-j\frac{1}{(\beta r)^3} + \frac{1}{(\beta r)^2}\right]e^{-j\beta r}$$

$$\overline{E}_\theta = \frac{\beta^2 \eta I_0 (dl) \sin\theta}{4\pi}\left[-j\frac{1}{(\beta r)^3} + \frac{1}{(\beta r)^2} + j\frac{1}{\beta r}\right]e^{-j\beta r}$$

$$\overline{H}_\phi = \frac{\beta^2 I_0 (dl) \sin\theta}{4\pi}\left[\frac{1}{(\beta r)^2} + j\frac{1}{\beta r}\right]e^{-j\beta r}$$

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The following observations are pertinent to these field expressions: **(a)** They satisfy all Maxwell's equations exactly. **(b)** For any value of r, the time-average value of the θ-component of the Poynting vector is zero, and the time-average value of the r-component of the Poynting vector is completely from the $1/r$ terms, thereby resulting in the time-average power crossing all possible spherical surfaces centered at the dipole to be the same. **(c)** At low frequencies such that $\beta r << 1$, the $1/(\beta r)^3$ terms dominate the $1/(\beta r)^2$ terms, which in turn dominate the $1/(\beta r)$ terms, and $e^{-j\beta r} \approx (1 - j\beta r)$ but the $1/(\beta r)^3$ terms dominate, thereby reducing the field expressions to the phasor forms of the quasistatic approximations given by (163) and (166). Finally, they are the familiar expressions obtained by using the magnetic vector potential approach.

Observations on the Field Solutions

1. They satisfy all Maxwell's equations exactly.
2. For any value of r, the time-average value of the θ component of the Poynting vector is zero, and the time-average value of the r component of the Poynting vector is completely from the $1/r$ terms, thereby resulting in the time-average power crossing all possible spherical surfaces centered at the dipole to be the same.
3. At low frequencies such that $\beta r << 1$, the $1/(\beta r)^3$ terms dominate the $1/(\beta r)^2$ terms, which in turn dominate the $1/(\beta r)$ terms, and $e^{-j\beta r} \approx (1 - j\beta r)$, thereby reducing the field expressions to the phasor forms of the quasistatic approximations.